The Problem of the Sun's Corner Altitude and Convergence of Fixed-Point Iterations in Medieval Indian Astronomy

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1 *Introduction*

A remarkable rule for calculating the altitude of the sun above the horizon first appeared in a Sanskrit treatise of the seventh century. As noted in [Pingree 1978, 571–2], it is an elaborately complicated formula which is applicable only under very restricted circumstances: when the sun is exactly at one of the four intercardinal directions, that is, when the angle between its vertical circle and the prime vertical circle passing through the east and west points is 45°. This condition gave to the problem the name *koṇaśaṅku*, 'corner gnomon' or 'corner altitude'.

As it happened, the demonstrably exact solution to this problem in the abovementioned seventh-century text was the prelude to the development of various approximate ones over the course of the next several centuries. These took the form of *asakṛt* ('not just once') or iterative rules, a very common type of computational technique in Sanskrit texts [Plofker 2002]. In particular, these approximations were of the sort now known as 'fixed-point' iterations, whose convergence can be quite unreliable—a fact that has helped shape entire subfields of modern mathematics, such as chaos theory. Using some of the mathematical tools now employed in these disciplines, we can see that Indian fixed-point iterations for the corner altitude were in fact susceptible to certain convergence problems, which were to some extent solved by later modifications. In this paper, we examine the mathematical behavior of these iterations, and reconstruct how some of their users apparently recognized and attempted to deal with the problems inherent in them.

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2 *The original rule for the sun's corner altitude*

The problem of finding the sun's altitude a above the horizon, given its declination δ, its corner direction d (measured southward from the east point, so that the southeastern d is 45°, the southwestern 135°, etc.) and the terrestrial latitude ϕ, is almost entirely peculiar to Indian astronomy.[1] Except in its level of difficulty, the corner altitude is typical of the problems in the Sanskrit astronomical topic known as 'Three Questions' (*tripraśna*), which deals with the various relationships between quantities determining direction, terrestrial location, and time: e.g., computing the local latitude from the noon equinoctial shadow of a gnomon, or the current time from the latitude and the sun's declination and altitude, etc. These are generally straightforward calculations based on the similarity of right triangles inside a hemisphere bounded by the celestial sphere and the observer's horizon. It is not certain what practical motivation (if any) inspired astronomers to tackle the esoteric problem of the corner altitude,[2] nor who first solved it successfully. The first known solution is attested in the chapter on 'Three Questions' of the *Brāhmasphuṭasiddhānta* (628 CE) of Brahmagupta, who states:

> Half the square of the radius, decreased by the square of the Sine of the sun's rising amplitude, multiplied by the square of twelve, is the 'First'. The 'Second' is the product of the Sine of the rising amplitude and twelve and the [noon] equinoctial shadow [s_0]. When the two are divided by the square of the equinoctial shadow added to 72, the square root of the First increased by the square of the Second, increased or decreased [when the sun is] in the northern or southern hemisphere [respectively] by the Second, is the Sine of altitude [for $d = 45°$].[3]

Brahmagupta says nothing about how this rule was derived or why it works. We shall explain its meaning in modern terms as follows. Figure 1 illustrates the corner altitude for a typical case

[1] See, however, [Kennedy and Debarnot 1979].

[2] Some *śilpaśāstra* or architectural texts mention the auspicious qualities of the various directions, but apparently without reference to the position of the sun.

[3] *Brāhmasphuṭasiddhānta* 3, 54–5 [Dvivedi, 66]. Here and in all subsequent quotations, square brackets are used to indicate clarifying interpolations that do not appear in the original Sanskrit.

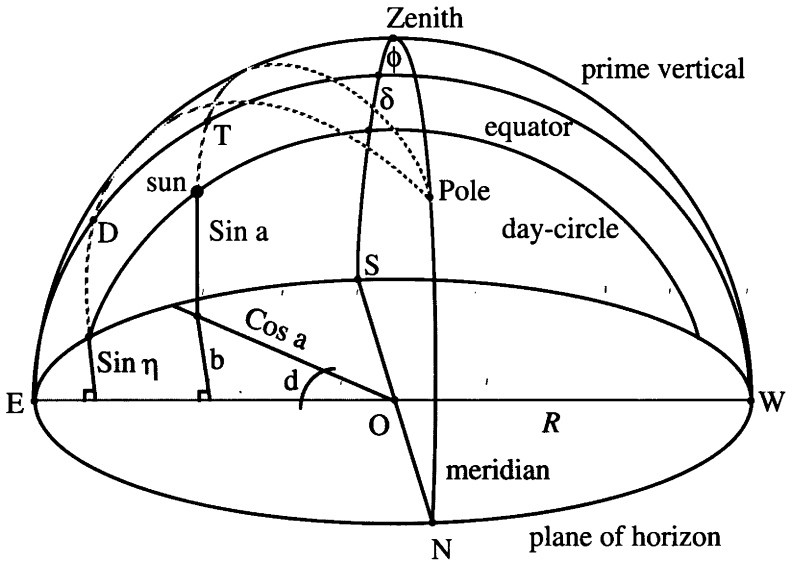

Figure 1: The celestial hemisphere above the horizon

in the northern hemisphere (only northern or positive latitudes are generally considered in Indian astronomy). An analytic solution can be derived geometrically from consideration of the quantities in the same three-dimensional picture and in its analemma (two-dimensional orthographic projection) in Figure 2, as follows: The sun's rising amplitude η, i.e., the arc between its rising-point and the east point E, is defined by

$$\frac{\text{Sin}\,\eta}{\text{Sin}\,\delta} = \frac{R}{\text{Cos}\,\phi}, \tag{1}$$

where the capitalized 'Sin', for instance, indicates the usual sine function scaled by some non-unity value of the trigonometric radius R. A southern declination is considered negative and a northern δ positive; we shall take the sign of η to correspond to that of δ, though Sanskrit texts treat its Sine as always positive and modify their addition and subtraction procedures depending on its direction. (In fact, they treat all Sines and Cosines as positive, but we shall follow modern practice in this regard.) The distance b between the east-west line EW and the point on the

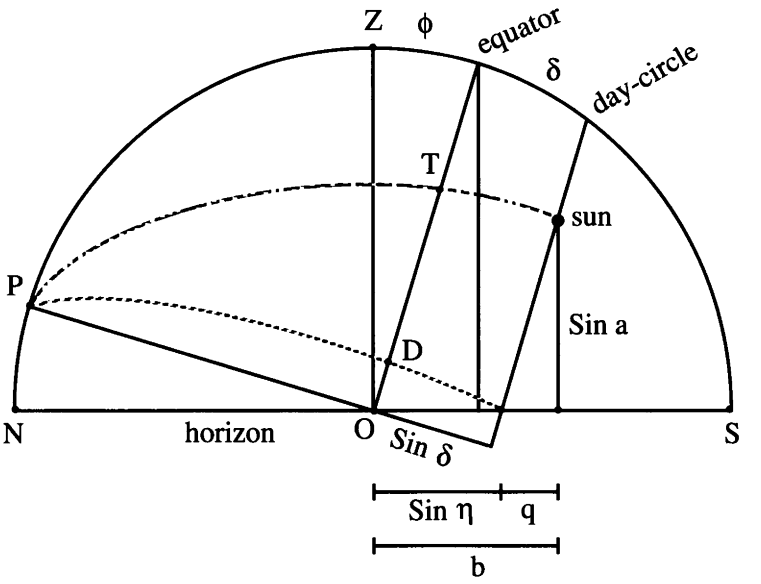

Figure 2: Analemma projection of celestial hemisphere

plane of the horizon directly beneath the sun is given by

$$\frac{b}{\operatorname{Cos} a} = \frac{\operatorname{Sin} d}{R}, \tag{2}$$

and we shall assume d and therefore b to be positive when (as is most usual in northern latitudes) the sun is south of the prime vertical. The solar altitude a can then be derived from these two quantities and the latitude ϕ as follows:

$$\frac{\operatorname{Sin} a}{b + \operatorname{Sin} \eta} = \frac{\operatorname{Cos} \phi}{\operatorname{Sin} \phi}. \tag{3}$$

Setting $R = 1$ so that we may use the familiar modern (lower-cased) trigonometric functions, we can reduce these rules to the expression

$$\frac{\sin a}{\cos a \sin d + \dfrac{\sin \delta}{\cos \phi}} = \frac{\cos \phi}{\sin \phi}. \tag{4}$$

This can be recast as a quadratic equation in terms of $\sin a$ as follows:

$$\sin^2 a(\tan^2 \phi + \sin^2 d) - \sin a \left(2\frac{\sin \delta}{\cos \phi} \tan \phi\right) + \left(\frac{\sin^2 \delta}{\cos^2 \phi} - \sin^2 d\right) = 0 \tag{5}$$

(and for simplicity's sake, we shall leave this and related expressions in terms of $\sin a$, rather than complicating the expressions further with inverse sines). The values of the root are found from the quadratic formula:

$$\sin a = \frac{\frac{\sin \delta}{\cos \phi} \tan \phi \pm \sqrt{\sin^2 d \tan^2 \phi - \sin^2 d \cdot \frac{\sin^2 \delta}{\cos^2 \phi} + \sin^4 d}}{\tan^2 \phi + \sin^2 d}. \tag{6}$$

The astronomical context constrains the type and sign of the roots by limiting the possible values of the parameters ϕ and δ. In the general quadratic equation $Ax^2 + Bx + C = 0$, neither of the roots $x = (-B \pm \sqrt{B^2 - 4AC})/2A$ can be real if $4AC > B^2$. Given A positive as in equation (5), if $C < 0$, both roots are real and of opposite signs; if $0 < 4AC < B^2$, both are real and have the same sign as $-B$. Substituting for A, B, and C in the above conditions the actual coefficients from equation (5)—and noting that ϕ is non-negative and δ must fall within the limits of the sun's yearly north-south motion in declination, which are considered in the Indian tradition to be $\pm 24°$—we can deduce the constraints on its roots as follows:

$$|\delta| \begin{cases} < |d| : \quad 1 - \dfrac{\sin^2 \delta}{\sin^2 d} & \begin{cases} > \sin^2 \phi & \Rightarrow \text{ 2 real roots of opposite signs} \\ < \sin^2 \phi & \Rightarrow \text{ 2 real roots of same sign as } \delta \end{cases} \\ > |d| : \quad \dfrac{\sin^2 \delta - \sin^2 d}{1 - \sin^2 d} & \begin{cases} > \sin^2 \phi & \Rightarrow \text{ no real root} \\ < \sin^2 \phi & \Rightarrow \text{ 2 real roots of same sign as } \delta \end{cases} \end{cases} \tag{7}$$

Physically, the absence of any real root $\sin a$ means that at the given latitude, the sun at the given declination never attains the direction either of d or of $-d$; two real roots of the same sign imply

that it reaches both d and $-d$ either above the horizon (when the roots are positive) or below the horizon (when they are negative); and if there are two real roots of opposite signs, the sun moves to d during the daytime and to $-d$ in the night, or vice versa. Since the sun's declination is considered as constant throughout a given day, its day-circle is symmetric about the prime meridian, so its altitude is the same at d and $(180 - d)$.

Because, as we stated above, d is an odd multiple of 45°, $|\sin d| = \sin 45° = 1/\sqrt{2}$; so equation (5) reduces to

$$\sin^2 a\,(\tan^2\phi + \frac{1}{2}) - \sin a\left(2\frac{\sin\delta}{\cos\phi}\tan\phi\right) + \left(\frac{\sin^2\delta}{\cos^2\phi} - \frac{1}{2}\right) = 0 \tag{8}$$

and equation (6) to

$$\sin a = \frac{\dfrac{\sin\delta}{\cos\phi}\tan\phi \pm \sqrt{\dfrac{1}{2}\tan^2\phi - \dfrac{1}{2}\dfrac{\sin^2\delta}{\cos^2\phi} + \dfrac{1}{4}}}{\tan^2\phi + \dfrac{1}{2}}. \tag{9}$$

With d thus restricted, $|\delta|$ is always less than $|d|$. Equation (7) then implies that there will be one real positive root $\sin a$ if and only if $\sin^2\phi < 1 - 2\sin^2\delta$; this is always true for any $\phi < 55°$ approximately, which includes all possible latitudes on the Indian subcontinent. So there should always be a unique positive real value for the corner altitude in our calculations.

Now we can rewrite the procedure from Brahmagupta's verses in algebraic notation, employing our sign convention for η defined above:

$$\mathrm{Sin}\, a = \sqrt{\frac{\left(\dfrac{R^2}{2} - \mathrm{Sin}^2\eta\right)\cdot 12^2}{72 + {s_0}^2} + \left(\frac{12 s_0 \cdot \mathrm{Sin}\,\eta}{72 + {s_0}^2}\right)^2} + \frac{12 s_0 \cdot \mathrm{Sin}\,\eta}{72 + {s_0}^2}. \tag{10}$$

Recall from equation (1) that $\mathrm{Sin}\,\eta = R \cdot \mathrm{Sin}\,\delta / \mathrm{Cos}\,\phi$; moreover, since the noon equinoctial shadow s_0 of a standard twelve-digit gnomon is also measured in digits, $s_0/12 = \tan\phi$ (see Figure 3). Given these identities, it is straightforward (though somewhat laborious) to show that Brahmagupta's rule—which became standard in later treatises—indeed reproduces the positive root from equation (9).

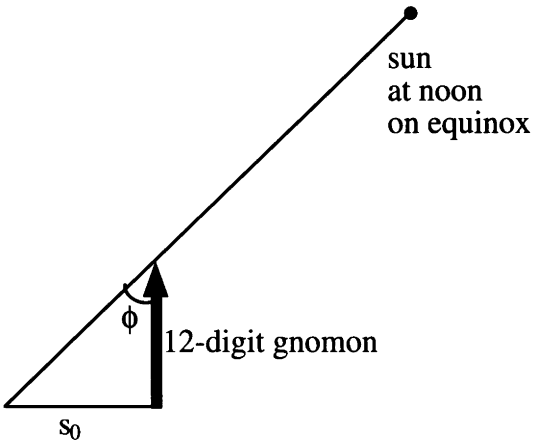

Figure 3: Noon equinoctial shadow

3 *A new solution by means of fixed-point iteration*

The first known iterative alternative to Brahmagupta's closed-form solution for $\sin a$ appeared a century or two after Brahmagupta's work, in the 'Three Questions' chapter of the *Śiṣyadhīvṛddhidatantra* of Lalla (which does not mention the closed-form version):

> When the square of the Sine of the sun's rising amplitude added to [some arbitrary] desired [amount, q_0] is multiplied by two and subtracted from the square of the radius, the square root [of that] is the Sin altitude in the corner direction. Then when [it] is multiplied by the noon equinoctial shadow and divided by twelve, the quotient is the [new] desired [quantity, q_1]. The Sine of the rising amplitude is increased by that, as before. Thus the iterative rule for the altitude [when] the sun is in the south. When [it is] in the northern hemisphere, this rule [is modified by using] the difference of the Sine of the rising amplitude and the desired [quantity, rather than their sum].[4]

In more general terms, and again employing our own sign convention for η,

$$\mathrm{Sin}\, a_0 \;=\; \sqrt{R^2 - 2(q_0 - \mathrm{Sin}\,\eta)^2},$$

[4] *Śiṣyadhīvṛddhidatantra* 4, 34–5 [Chatterjee, I, 72].

$$q_1 = \operatorname{Sin} a_0 \cdot \frac{s_0}{12} \qquad \left[= \operatorname{Sin} a_0 \cdot \frac{\operatorname{Sin}\phi}{\operatorname{Cos}\phi}\right],$$
$$\operatorname{Sin} a_1 = \sqrt{R^2 - 2(q_1 - \operatorname{Sin}\eta)^2}, \tag{11}$$

and so on until the true value of $\operatorname{Sin} a$ is reached. Here Lalla introduces a new quantity that we shall call q, which the successive 'desired amounts' q_i are supposed to approximate. Evidently, q is the sum of b and $\operatorname{Sin}\eta$: for it is clear from Figure 1that when $d = 45°$, $\operatorname{Cos}^2 a = 2b^2$, and we see from similar right triangles in Figure 2that the sum of (negative) $\operatorname{Sin}\eta$ and (positive) b is indeed equal to $\operatorname{Sin} a \cdot \operatorname{Sin}\phi/\operatorname{Cos}\phi$. In physical terms, the user is being asked to guess the distance b between the east-west line and the point directly beneath the sun; then, assuming that that point lies in the corner direction (which is usually astronomically impossible for the estimated b at the given ϕ and δ), to calculate its height; and, supposing that altitude to be on the given day-circle, to calculate a new distance b, and so on. But will the mere repetition of these computations starting with some arbitrarily guessed q_0 in place of the required sum $b + \operatorname{Sin}\eta$ actually come up with the correct answer, and if so, why?

To see the mathematical implications of this procedure, let us consider the modern definition of a fixed-point iteration.[5] Briefly, when a root r of a function $f(x)$ is sought, a fixed-point iteration finds it by employing some auxiliary function $g(x)$ such that the desired root of f is also a fixed point $r = g(r)$ of g. This fixed point is found by choosing some initial value or 'seed' r_0 and then computing the successive values $r_1 = g(r_0)$, $r_2 = g(r_1)$, and so forth. If the iteration is convergent, these successive values (the so-called 'orbit' of the seed r_0) will approach closer and closer to the 'attracting fixed point' r, which falls at the intersection of the graph of the auxiliary function $y = g(x)$ and the straight line $y = x$, as illustrated in Figure 4. On the other hand, if the iteration does not converge, the successive r_i may move farther and farther away from r, which in this case is called a 'repelling'

[5] A more detailed treatment of this topic can be found in [Devaney 1992]. I thank Jared Herzberg for providing the reference, and Davide P. Cervone and Homer White for many helpful comments on this section. The calculations for the iterative functions were carried out in Waterloo Maple V, and their orbit diagrams were plotted using John Hubbard's and Beverly West's Analyzer 9.0.

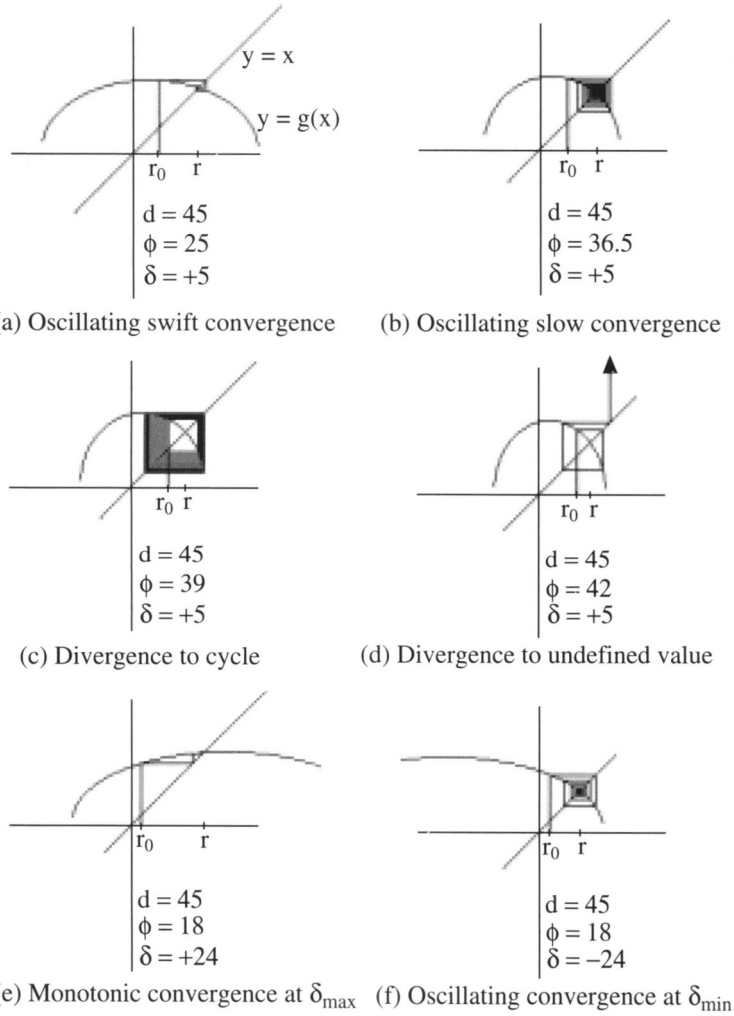

Figure 4: Orbits of $g(\sin a)$ for various ϕ and δ

fixed point; or the r_i may approach a 'cycle' or oscillation between other values at a finite distance from r, which is then said to be 'neutral', neither attracting nor repelling.

Finding an auxiliary function $g(x)$ that has a fixed point where $f(x)$ has a root is generally just a matter of algebraic reshuffling of the terms of f; however, finding such a $g(x)$ that will converge reliably can be more difficult. The behavior of convergent and non-convergent iterations is demonstrated by the sample orbit diagrams of Figure 4, showing various forms of an iterative function $g(x)$ (defined in the following section) whose graph in each case is a semi-ellipse. In these diagrams, the initial value of the function at the seed r_0 falls on the vertical line extending from the x-axis at r_0 up to the graph of $g(x)$. Then a horizontal line is drawn from that point $(r_0, g(r_0))$ over to the point $(g(r_0), g(r_0)) = (r_1, r_1)$ on the line $y = x$. A vertical line is then extended from that point to the point $(r_1, g(r_1))$ on the graph of $g(x)$, and the process continues as the orbit 'walks' toward or away from the fixed point r. As shown in the figure, the 'steps' of the orbit may approach or recede from the fixed point from one side only (monotonic convergence or divergence, when the slope of the curve at the fixed point is positive) or may jump between too-small and too-large values (oscillating convergence or divergence at a negative slope).

The pictured orbit diagrams also suggest that in a convergent iteration, the differences between the successive horizontal steps get smaller and smaller as the size of the steps themselves approaches zero; if, on the other hand, the successive differences increase, or the step size approaches some finite non-zero amount, or both, then the iteration will not ultimately converge. For smooth and well-behaved functions such as the ones we shall deal with here, this is a reasonably accurate statement of the convergence condition.

Somewhat more formally, we say that convergence for all possible seeds in some interval around r requires that for all the r_i in the orbit of a seed r_0 in that interval,

$$|r_i - r_{i+1}| > |g(r_i) - g(r_{i+1})| \qquad \text{or} \qquad \left| \frac{g(r_i) - g(r_{i+1})}{r_i - r_{i+1}} \right| < 1.$$

This is roughly equivalent to requiring that the absolute value of the first derivative g' of the function at r must be strictly less

than 1. (For a more intuitive justification, recollect that the first derivative is just the slope of the tangent line to the function at that point. Since each successive function value $g(r_i)$ becomes the next input r_{i+1} to the function, a continual decrease in the size of the step-differences means that each vertical step is smaller than the preceding horizontal step: i.e., the y-values change more slowly than their corresponding x-values, so the absolute value of the slope is less than 1.) When $|g'(r)| > 1$, the iteration will diverge; for a neutral fixed point, $|g'(r)| = 1$. Values of $|g'(r)|$ close to 1 mean that the iteration will converge or diverge slowly; the speed of convergence increases as the derivative approaches zero. (When the first derivative at r actually is zero, i.e., the tangent line to the curve at that point is horizontal, the order of convergence is quadratic instead of linear, and the fixed point is called 'superattracting'.)

4 *Evaluating Lalla's fixed-point technique*

We can now return to Lalla's iterative rule for the corner altitude, as represented by equation (11), and assess how effective it is according to the analysis of fixed-point algorithms presented above. We wish to know, first: is it in fact a valid auxiliary function with respect to the earlier closed-form solution, that is, does it have a fixed point where the quadratic in equation (8) has a root? And second: if so, does the iteration actually converge to that fixed point, and how quickly? We start by rewriting Lalla's procedure as a single iterative equation in terms of $\sin a$ with modern trigonometric functions:

$$\sin a_{n+1} = g(\sin a_n) = \sqrt{1 - 2\left(\sin a_n \tan\phi - \frac{\sin\delta}{\cos\phi}\right)^2}. \qquad (12)$$

We solve for the fixed point $\sin a = g(\sin a)$ by setting $\sin a_{n+1} = \sin a_n = \sin a$, and can show after some algebraic manipulation that it occurs at

$$\sin a = \frac{2\frac{\sin\delta}{\cos\phi}\tan\phi \pm \sqrt{2\tan^2\phi - 2\frac{\sin^2\delta}{\cos^2\phi} + 1}}{2\tan^2\phi + 1}, \qquad (13)$$

which is evidently (after canceling 2's) exactly the same as the expression for the roots of the quadratic in equation (9). Further-

more, taking a (positive) square root at every iteration means that all the successive $\sin a_n$ will be positive. So the fixed point of Lalla's iteration is indeed identical to the positive real root of Brahmagupta's quadratic.

The question of whether the iteration will actually converge to this fixed point is somewhat more complicated. Recall that convergence is dependent on the size of the first derivative of the function at the fixed point, which is dependent on the values of the terrestrial latitude ϕ and the solar declination δ. Strictly speaking, both the quadratic equation (8) and the iterative equation (12) represent not individual functions but families of functions, parametrized by the constants ϕ and δ. We have now shown that the fixed points of the iterated function will always be the same as roots of the corresponding quadratic; but the orbits produced by the iteration will change with the values of these two parameters. We must therefore modify our question and ask which values of ϕ and δ, if any, will produce an iteration with an attracting fixed point.

Differentiating the iterative function in equation (12) with respect to $\sin a$ gives

$$g'(\sin a) = \frac{-2\tan\phi\left(\sin a\tan\phi - \dfrac{\sin\delta}{\cos\phi}\right)}{\sqrt{1-2\left(\sin a\tan\phi - \dfrac{\sin\delta}{\cos\phi}\right)^2}}. \tag{14}$$

To satisfy our convergence condition, the absolute value of this first derivative at the fixed point must be less than 1. So setting $|g'(\sin a)| < 1$, and substituting into the above expression for g' the value of the positive fixed point $\sin a$ from equation (13), we find that the absolute value of the quantity

$$\frac{-2\tan\phi\left(\dfrac{2\dfrac{\sin\delta}{\cos\phi}\tan\phi + \sqrt{2\tan^2\phi - 2\dfrac{\sin^2\delta}{\cos^2\phi}+1}}{2\tan^2\phi+1}\cdot\tan\phi - \dfrac{\sin\delta}{\cos\phi}\right)}{\sqrt{1-2\left(\dfrac{2\dfrac{\sin\delta}{\cos\phi}\tan\phi + \sqrt{2\tan^2\phi - 2\dfrac{\sin^2\delta}{\cos^2\phi}+1}}{2\tan^2\phi+1}\cdot\tan\phi - \dfrac{\sin\delta}{\cos\phi}\right)^2}}$$

must be strictly less than 1.

It is not intuitively obvious from this untidy expression which particular values of ϕ and δ will satisfy the required condition. In broad terms, though, using $\delta < 0$ lowers the maximum ϕ for which the inequality will hold, and using $\delta > 0$ raises it. (The reason is illustrated by the orbit diagrams shown for different values of ϕ and δ in Figure 4: in general, increasing ϕ increases the second derivative or rate of change of the slope of the function, making a 'steeper' semi-elliptical arc, while decreasing δ shifts its midpoint (at which the slope is zero) towards the left, meaning that the curve then intersects the $y = x$ line at a point where its slope is steeper. Increasing δ shifts the graph toward the right, with the opposite effect.) More precisely, we can set δ equal to some chosen value and then solve the inequality for the corresponding maximum ϕ. We find[6] that when $\delta = 24°$ at the summer solstice, the upper bound on ϕ is about 49°, for $\delta = 0$ at equinoxes it is about 35°, and for $\delta = -24°$ it is about 21.5°.

Therefore, an astronomer using Lalla's iterative technique to compute the southeast *koṇaśaṅku* at, say, Ujjain ($\phi \approx 23°$) on the day of the summer solstice, using a standard Sine table with $R = 3438$ and initially guessing q_0 to be 1000, would find the approximate results (3359, 3435, 3437, 3437)—a short sequence converging quickly to the same value of $\mathrm{Sin}\, a$ that Brahmagupta's closed-form solution produces. But the same astronomer performing the same calculation with, for example, $q_0 = 500$ at the winter solstice would come up with the successive values 1915, 971, 2088, 501, 2414, and the square root of -1125271, a bewildering and completely useless result. Moreover, since the speed of convergence decreases as the decreasing δ lowers the upper bound on ϕ, even iterations that ultimately do converge to a fixed value can require long and wearisome toil in order to get there. Our hypothetical Ujjain astronomer at an equinox, for example, starting with $q_0 = 1000$, would have to plod through the computation of 3134, 2878, 2972, 2939, 2951 and 2946 before eventually coming

[6] The expression used in Maple to produce the upper bound on ϕ for, e.g., $\delta = 24°$ is as follows: `solve (1 = abs( subs( d = (24 *3.14159 / 180), subs( x=( (2 * tan(f) * (sin(d) / cos(f)) + sqrt(2 * tan(f)^ 2 - 2 * (sin(d)^ 2 / cos(f)^ 2) + 1))/ (2 * tan(f)^ 2 + 1) ), ( (-2 * tan(f)*( tan(f) * x - (sin(d) / cos(f)) ))/ sqrt(1 - 2 * (tan(f) * x - (sin(d) / cos(f)))^ 2) )) ) ), f).`

to rest at $\mathrm{Sin}\, a = 2948$. At $\delta = -10°$ he would need to perform some fifteen iterations to get a result precise to the nearest integer, and at $\delta = -20°$ he would require more than ninety before settling into a cycle between two repeating values and never reaching the true $\mathrm{Sin}\, a$ at all.

It is thus clear that the ingenious method of Lalla (or his unidentified source) for the corner altitude, although procedurally somewhat simpler than Brahmagupta's rule, is liable to crippling malfunctions. Yet Lalla's text gives no clue that he was aware of their existence, although since his own local latitude was probably 21° or thereabouts,[7] he could hardly have avoided encountering some of them if he used the method frequently. Various later authors also discussed the same rule without offering any caveats: e.g., Govindasvāmin (9th c., $\phi \approx 10°$?), Vaṭeśvara (c. 904, $\phi = 23;45°$), and Lalla's own commentator Bhāskara II (b. 1114, $\phi \approx 24°$).[8] Apparently these astronomers (at least the ones at higher latitudes) either never made much use of this iterative rule, or else discovered some of its shortcomings but deliberately refrained from commenting on them. The syncretic nature of Sanskrit astronomy lends credence to both explanations: respect for earlier authorities and appreciation of computational ingenuity for its own sake, combined with the rather free organization of the texts, permitted the survival of many techniques that had little practical value, as well as some that were known to be inaccurate. When we consider that Lalla's rule requires computing one Sine and one square root at every iteration, whereas Brahmagupta's requires only one of each to produce the exact result, it is tempting to conclude that the iterative formula was probably preserved more as an interesting mathematical curiosity than as part of the working toolkit of practicing astronomers.

5 *Evidence for recognition of the method's failure*

Hints in some later texts, however, strongly suggest that the convergence problems with the *koṇaśaṅku* rule were in fact noticed,

[7] For the little that is known about Lalla's place of origin and his life in general, see [Pingree 1970, V, 545] and [Chatterjee, II, xv].

[8] These references occur in the commentary on *Mahābhāskarīya* 3, 41 [Kuppanna Sastri, 155–8]; *Vaṭeśvarasiddhānta* 3, 12, 3–4 [Shukla, I, 204–9]; and *Siddhāntaśiromaṇi* Gaṇita 3, 30 [Śāstrī, 75–7].

and that partially successful methods were developed to cope with them. (Of course, these developments were unrelated to the modern tools and concepts we used in the previous sections to investigate convergence issues: *asakṛt* rules for the Indian astronomer were a matter not of graphs, functions, and derivatives, but merely of successive answers produced by repeated calculations.) What may be the first such hint is seen in the *Siddhāntaśiromaṇi* of Bhāskara II, where in presenting the same iterative rule, he remarks:

> When one has subtracted the square of the Sine of the rising amplitude, multiplied by two, from the square of the radius, that square-root is now the 'corner gnomon'...
>
> [Commentary:] Here, because of [our] ignorance of the 'corner gnomon', there is [also] ignorance of [the distance of] the base of the 'gnomon' [from the east-west line, i.e., b]. Only the Sine of the rising amplitude is known. That is initially considered [as] the segment [b]. Hence the 'corner gnomon' resulting from [the rule] beginning 'the square of the Sine of the rising amplitude, multiplied by two, from the square of the radius...' is approximate. Then, by means of the rule repeatedly [applied], it becomes correct.[9]

The seed q_0 is here specified as zero, meaning that b when $d = 45°$ is initially guessed to be the same as $\operatorname{Sin}\eta$. Obviously, since the sun's daily motion always—except for observers at the equator—takes it south of its rising amplitude, a small nonzero q_0 would be a better estimate. Though this use of a zero seed seems counterproductive, it may possibly have originated as a diagnostic technique. For it is evident from equation (11) that for negative δ, when q_0 is as small as possible (i.e., zero), $\operatorname{Sin} a_0$ and therefore q_1 will be as large as possible. If this q_1 does not cause the iteration to fail by producing a negative number under the square-root sign in the computation of $\operatorname{Sin} a_1$, the user can be confident that no subsequent (smaller) q_n will do so, and thus the iteration will at least not diverge to an undefined value.

This notion of the zero seed as a diagnostic test for divergence remains speculative, since Bhāskara does not describe it as such; he seems rather to be suggesting that we are forced to approximate our first b by $\operatorname{Sin}\eta$ since 'only $\operatorname{Sin}\eta$ is known'. Later in the

[9] *Siddhāntaśiromaṇi* Gaṇita 3, 30 [Śāstrī, 75–7].

same comment he does enumerate the various directions in which the *koṇaśaṅku* may appear at latitudes of 55° or more, but does not illustrate the use of the iterative rule in such cases or mention the possibility of its failure.

A different and more constructive modification of Lalla's original rule appears in a commentary on Lalla's work by Mallikārjuna Sūri, who worked c. 1178 at a latitude of approximately 18° [Chatterjee, II, xxiii–xxvi]. Mallikārjuna's remarks on the relevant verses of the *Śiṣyadhīvṛddhidatantra* run thus:

> The Sine of the sun's declination, multiplied by the equinoctial hypotenuse, divided by twelve, is the Sine of the sun's rising amplitude. The square of that, multiplied by two, is subtracted from the square of the radius. The square-root of that, multiplied by the noon equinoctial shadow and divided by twelve, is the first desired [quantity, q_1]. That quantity, when the declination is south, is added to the Sine of the sun's rising amplitude, or subtracted from the Sine of the sun's rising amplitude when the declination is north. If the desired [quantity] is greater than the Sine amplitude, [use] their difference. Then, when one has multiplied the square of that by two, it is subtracted from the square of the radius. The square-root of that is the corner altitude.
>
> Having multiplied that by the equinoctial shadow, one should divide it by twelve. The quotient is the second desired [quantity, q_2]. When one has considered half the sum of these second and first quantities as the desired quantity, as before it is to be added to the Sine of the sun's rising amplitude when [the sun is] in the south, [or] subtracted in the north. Then when one has multiplied the square of that by two and subtracted it from the square of the radius, the square-root is the corner altitude. After multiplying that separately by the noon equinoctial shadow, one should divide it by twelve. The quotient is a [new] desired [quantity, q_3]. As before, when one has corrected that quantity by the Sine of the rising amplitude, multiplied its square by two, [and] subtracted it from the square of the radius, the square-root [of the result] is the corner altitude. [Computing] by iteration in this way again and again, the corner altitude is determined. When it is equal to [the value from the previous [iteration], the corner altitude is accurate.[10]

Here, Mallikārjuna has worked into his explication of Lalla's rule (again, without articulating any criticism of it) a significant alter-

[10] Comm. on *Śiṣyadhīvṛddhidatantra* IV, 34–5 [Chatterjee, I, 73].

ation.[11] His verbal procedure can be recast symbolically in our usual way as follows:

$$\begin{aligned}
\operatorname{Sin} a_0 &= \sqrt{R^2 - 2(0 - \operatorname{Sin}\eta)^2} \qquad [q_0 = 0], \\
q_1 &= \operatorname{Sin} a_0 \cdot \frac{s_0}{12} \qquad \left[= \operatorname{Sin} a_0 \cdot \frac{\operatorname{Sin}\phi}{\operatorname{Cos}\phi}\right], \\
\operatorname{Sin} a_1 &= \sqrt{R^2 - 2(q_1 - \operatorname{Sin}\eta)^2}, \\
q_2 &= \operatorname{Sin} a_1 \cdot \frac{s_0}{12}, \\
\operatorname{Sin} a_2 &= \sqrt{R^2 - 2\left(\frac{q_1 + q_2}{2} - \operatorname{Sin}\eta\right)^2}, \\
q_3 &= \operatorname{Sin} a_2 \cdot \frac{s_0}{12}, \\
\operatorname{Sin} a_3 &= \sqrt{R^2 - 2\,(q_3 - \operatorname{Sin}\eta)^2}.
\end{aligned} \tag{15}$$

The first four steps of the algorithm obviously mirror those of Lalla in equation (11), except that like Bhāskara, Mallikārjuna uses a zero q_0. But the next step deviates from the original procedure, being equivalent to the expression

$$\sin a_2 = \sqrt{1 - 2\left(\frac{\sin a_0 + \sin a_1}{2}\tan\phi - \frac{\sin\delta}{\cos\phi}\right)^2}.$$

In other words, Mallikārjuna recommends finding the third approximation to $\sin a$ by plugging into the iterative equation not the second approximation, as Lalla directs, but the mean of the first and second approximations. Iterating this step is equivalent to defining a new iterative function which we shall call $\gamma(\sin a)$:

$$\sin a_{n+1} = \gamma(\sin a_n) = \frac{\sin a_n + g(\sin a_n)}{2} \tag{16}$$

It is clear from the form of the above expressions that $\gamma(\sin a)$ has the same fixed point, and is defined over the same domain,

[11] It is quite possible, of course, that Mallikārjuna's modifications are not originally due to Mallikārjuna, nor Lalla's rule to Lalla, for that matter. The dearth of specific attributions in texts like these, and the vast number of such texts that remain unpublished, mean that the ultimate origins of these and similar innovations remain very doubtful. The fact that Mallikārjuna's commentary on this rule immediately goes on to describe Brahmagupta's closed-form solution as an alternative *may* imply that he was personally aware of the problems of the iterative approach, but then again it may not.

as $g(\sin a)$ in equation (12). But its first derivative is different:

$$\gamma'(\sin a) = \frac{1}{2} + \frac{g'(\sin a)}{2}. \tag{17}$$

Evidently, γ will pass the first-derivative test for an attracting fixed point in more cases than g will—specifically, $|\gamma'| < 1$ when $-3 < g' < 1$.[12] This implies that at $\delta = -24°$, the fixed point of the function γ is attracting for ϕ up to about 33°.

Also, this new method often converges more quickly than the original one. This makes intuitive sense when we consider an orbit oscillating between too-small and too-large approximations to the fixed point: splitting the difference between two successive values generally cuts down on the size of the oscillation. More precisely, a comparison of the absolute values of the first derivative tells us that

$$|\gamma'| \begin{cases} < |g'| & \text{for} \quad -1 < g' < -1/3 \\ > |g'| & \text{for} \quad -1/3 < g' < 1. \end{cases}$$

Recall that even the largest positive δ does not shift the graph of g very far to the right of the y-axis, so the greatest positive slope ever actually attained at the fixed point will be little larger than zero. So in most of the possible cases where the iteration of g will converge, that of γ will converge faster, as in Figure 5; and it will also converge in some cases where g does not.

Unfortunately, the latter advantage is largely nullified by Mallikārjuna's requiring a seed of zero, which, as discussed above, will cause an immediate failure for any iteration of Lalla's algorithm that would ultimately diverge. Thus a user at higher latitudes, working with values of ϕ and δ for which g would diverge but γ converge, would see his iteration crash in the computation of $\operatorname{Sin} a_1$ before he could apply γ. At Mallikārjuna's own latitude of about 18°, however, Lalla's original iteration g is everywhere convergent, so this would not be a problem for him.

6 *The generalization of Lalla's rule by Parameśvara*

A new refinement of the iterative rule for the corner altitude emerged in the work of Parameśvara (c. 1400, $\phi = 10;51°$), the

[12] I'm indebted to John Feroe for pointing out this feature, and for the following explanation of it.

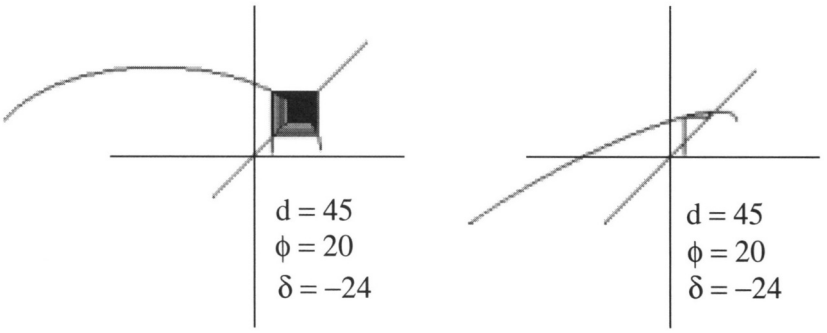

Figure 5: Orbits of $g(\sin a)$ and $\gamma(\sin a)$ for identical ϕ and δ

student of the famous leader of the Kerala school in Indian astronomy and mathematics, Mādhava of Saṅgamagrāma [Pingree 1970, IV, 187–92]. Parameśvara was familiar with Lalla's iteration, having written a supercommentary on the work of Govindasvāmin that included a version of it. In this supercommentary, he also presented a generalized form of this rule for arbitrary d, that is, an iterative solution to the quadratic in equation (5):[13]

> The determination of the '[great] shadow' [Cos a] and so forth should be [made] not only when the sun is standing in a corner direction, but it is computed also when [the sun is] in [any] desired direction. How? It is said:
>
> When one has estimated a desired 'gnomon-distance' [q], the segment [equal to the distance between the base of the altitude and the east-west line, b] is [found] from the sum or difference of that and the Sine of the rising amplitude. The segment [b], multiplied by the radius and divided by the Sine of the desired [azimuth] dependent on the radius, is the 'Sine of visibility' [Cos a]. The square root of the difference of the squares of the radius and that [Cos a] is the 'gnomon' [Sin a]. The 'gnomon' is multiplied by the Sine of the latitude and divided by the Cosine of the latitude. And that is [a new] 'gnomon-distance' [q]. ¿From the sum or difference [of that] and the Sine of the rising amplitude the

[13] Interestingly, neither Parameśvara nor anyone else seems to have sought a non-iterative solution by generalizing Brahmagupta's closed-form rule itself.

> [new] segment [b], and then in the same way the 'Sine of visibility' and so forth, [are computed]. And having made them again, one should [continue to] iterate.[14]

Rewriting this symbolically,

$$\begin{aligned} \operatorname{Cos} a_0 &= \frac{(q_0 - \operatorname{Sin}\eta)\cdot R}{\operatorname{Sin} d}, \\ \operatorname{Sin} a_0 &= \sqrt{R^2 - \operatorname{Cos}^2 a_0}, \\ q_1 &= \operatorname{Sin} a_0 \cdot \frac{\operatorname{Sin}\phi}{\operatorname{Cos}\phi}, \\ \operatorname{Cos} a_1 &= \frac{(q_1 - \operatorname{Sin}\eta)\cdot R}{\operatorname{Sin} d}. \end{aligned} \tag{18}$$

This expanded rule (which may be of Parameśvara's own devising—he does not discuss its origin) can easily be recast as a generalized version of equation (12):

$$\sin a_{n+1} = g(\sin a_n) = \sqrt{1 - \frac{1}{\sin^2 d}\left(\sin a_n \tan\phi - \frac{\sin\delta}{\cos\phi}\right)^2}. \tag{19}$$

The value of the fixed point $\sin a$ and the form of the first derivative $g'(\sin a)$ are identical to the expressions in equations (13) and (14) respectively, except that all the factors of 2 are replaced by the more general $1/\sin^2 d$. Therefore the fixed point of the new iteration will again be the same as a root of the corresponding quadratic—in this case, equation (5)—but as indicated by equation (7), the existence of a unique real positive root is no longer guaranteed.

Even when a real positive $\sin a$ exists, the iteration of this generalized $g(\sin a)$ will not always converge to it. The factor of $1/\sin^2 d$ in our new version of g' means that $|g'|$ will tend to exceed 1 as d gets smaller, so the upper bound on ϕ for which the iteration for a given δ will fail decreases with d. For example, when $d = 22.5°$ and $\delta = -20°$, the constraints in equation (7) imply that there will be one positive value of $\sin a$ for any $\phi < 26°$. But as illustrated in Figure 6, the generalized iterative equation fails to converge to that value for ϕ as small as 10°. In fact, at

[14] *Siddhāntadīpikā* on *Mahābhāskarīya* III, 41 [Kuppanna Sastri, 158–9].

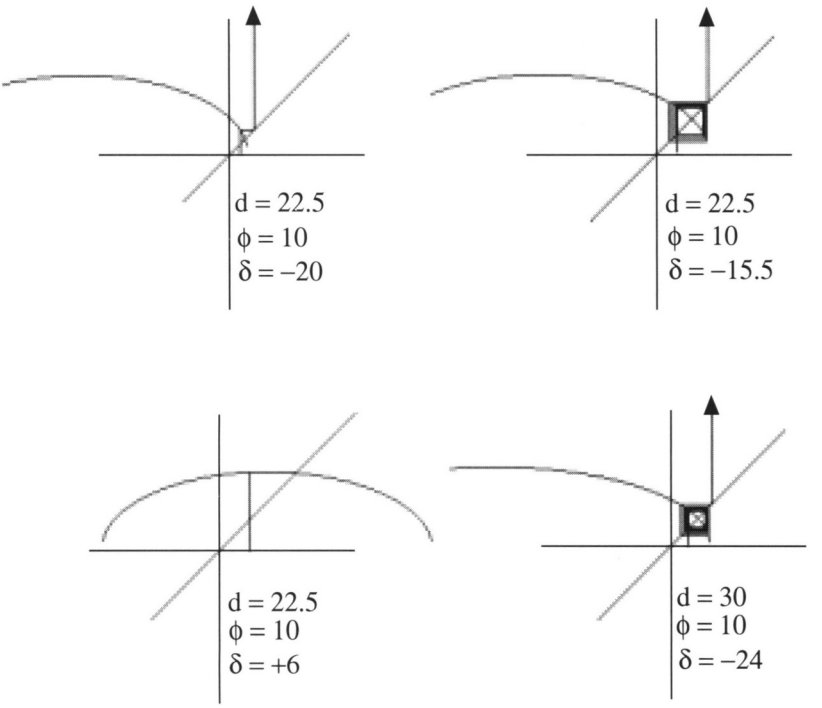

Figure 6: Orbits of generalized $g(\sin a)$ with $d < 45°$

that latitude, no iteration with $d = 22.5°$ will converge unless δ is greater than about $-15.5°$, and convergence remains sluggish as long as δ is less than $-12°$ or so. So even Parameśvara at his low latitude on the southwestern coast must have run into trouble if he tested the rule for sufficiently small d and negative δ—though, as usual, we see no allusion in the text to any such difficulty.

7 *Parameśvara's new iterative method for the* koṇaśaṅku

Nonetheless, there is reason to suspect that Parameśvara eventually abandoned this iterative approach precisely on account of its convergence problems. For in another (probably subsequent) work, the *Goladīpikā* [Pingree 1970, V, 188–9], he again addresses

the problem of the sun's corner altitude, but does not refer to the now-standard technique of Lalla. Instead, he prescribes the following more complicated rule (with his own commentary on the verses):

> An iterative method should be performed with intelligence for the sake of finding the corner altitude. [When the sun is] in the south, if the 'leg' is small[er than the 'arm'], something should be subtracted from the *ghaṭikās* [of time] past [since sunrise] or to come [till sunset]. If the 'arm' in that case is small[er], something should be added to the past or future *ghaṭikās*. When the sun stands in the north and moves only on that [northern] side [of the prime vertical circle], [the same corrections should be made] in that way. If the motion is on both [sides of the prime vertical circle], [first make the correction] as before to obtain a southern Sine altitude, [and then make the corrections] in reverse to obtain the northern Sine altitude. Here, the amount to be subtracted etc. is to be guessed by means of [one's] intelligence. By others, the difference between the 'arm' and the 'leg' is subtracted from or added to [the time] in *asus*. [Repeat the procedure] in just this way.
>
> [Commentary:] Whenever the 'arm' [b] and 'leg' [the east-west distance between the end of the segment b and the north-south line] of the 'shadow'-hypotenuse [Cos a] are equal, then the sun has arrived at a corner direction. Therefore the 'gnomon' [or height, Sin a] observed at that time is the 'corner gnomon'. When one has produced equality [between] the 'arm' and 'leg' by an iterative rule, the corner altitude is determined. He [i.e., Parameśvara himself] states the iterative method [in the lines beginning] '[When the sun is] in the south...'. When the sun is in a direction south of the east-west line and a southern corner altitude is to be found, if the 'leg' is smaller than the 'arm' of the great 'shadow'-hypotenuse attained at a given [amount of] past or future *ghaṭikās*, some amount of time should be subtracted from the past or future *ghaṭikās*. In that case, the sun is south of the corner: that is the meaning. Likewise, if the 'arm' is smaller than the 'leg', some amount of time is added to the past or future *ghaṭikās*. Thus the rule [when the sun is] in the southern hemisphere. If the sun, standing in a northern direction, moves only in the northern part [of the sky], then for the sake of determining the Sine altitude standing in a northern [corner] direction, the rule is [applied] just in that [same] way. And if the sun in the northern hemisphere moves to a southern [corner] direction too, then in order to determine the Sin altitude in the southern

> [corner] direction, the subtraction and addition are [prescribed] as before. Then in order to determine the northern Sine altitude, the subtraction and addition are reversed. Here the amount of the subtracted or added [quantity] is to be guessed with intelligence. Otherwise, *asus* measured by the difference of the 'arm' and 'leg' are subtracted from or added to the past or future *asus*. One should do [this] in the same way again until there is equality [between] the 'arm' and 'leg'. Then the iterated altitude becomes the corner altitude.[15]

The *ghaṭikā* is a time-unit equal to one-sixtieth of a day, and an *asu* is 1/360 of that, or four seconds; if a day is taken to equal 360° of revolution of the celestial equator, then an *asu* is the time it takes for one minute of arc to pass a given point. (Since $R = 3438 \approx 360 \cdot 60/2\pi$, *asus* are like our modern radians in allowing sines and angles to be expressed in the same units.) Sanskrit mathematical texts generally distinguish between the two legs of a right triangle with terms translated here as 'arm' (referring in this case to our b) and 'leg' (the other leg of the right triangle containing b and the hypotenuse or 'great shadow' $\mathrm{Cos}\, a$).

Here, Parameśvara is recommending that the user compare the sizes of b and the 'leg' $\sqrt{\mathrm{Cos}^2 a - b^2}$ at a particular moment in time: when the sun is in a corner direction and the hypotenuse $\mathrm{Cos}\, a$ makes a 45° angle with both, they will be equal. If they are not equal at the chosen moment, one must adjust the time till they become so. That is, if the 'leg' is longer than b, the sun is north of its corner location, and the time t since sunrise (or till sunset, in the case of a western *koṇaśaṅku*) must be increased; if b is longer, the sun is south of the corner position and t must be decreased. So the new time estimate should be the sum of t and the positive or negative quantity $\sqrt{\mathrm{Cos}^2 a - b^2} - |b|$. (As Parameśvara notes, however, the sign of the correction must be reversed for the first of two corner positions occurring in the same half-day. As noted in the discussion of equations (7)–(9) above, this is not possible at latitudes lower than about 55°; so for the present, we shall neglect this possible application of Parameśvara's rule and concentrate on explaining how it works for more typical corner altitudes.)

[15] *Goladīpikā* 4, 15–18ab [Sarma, 47].

Since Parameśvara does not explain in detail how Sin a and b are to be found from t, we shall use in our analysis his worked example from a subsequent verse and its commentary:

> [When] the true sun stands in the middle of Aries, ... the Sine altitude in the direction of Agni [i.e., south-east] is to be stated, and [the Sine altitude] standing in the middle [between] the directions of Agni and Indra [east]. The latitude-shadow [Sin ϕ] is 647.
>
> [Commentary:] He states an example [beginning] '[When] the true sun...' When the true sun is at the middle of Aries, ... the Sine altitude of the sun at the corner [direction] at that time is to be stated, and the Sine altitude of the sun at the middle [between the directions] of Agni and Indra is to be stated [as well]... These Sine altitudes are to be stated [for] the place where Sine latitude is equal to 647; this is the statement of the example. Here the Sine latitude is equal to 647; therefore the Cosine latitude is determined [as] equal to 3377. Because of that [longitude of] the sun, the Sine of the rising amplitude is determined [to be] equal to 368...
>
> For the sake of [computing] the corner altitude, the elapsed *ghaṭikās* of the day are considered [to be] 8. [On computing] with those, the Sine altitude is determined [to be] equal to 2516. The 'gnomon-distance' [q] is equal to 482. Its 'shadow' [Cos a] is equal to 2343; that is the hypotenuse. Here the Sine of the rising amplitude is equal to 368. The difference of the gnomon-distance and the Sine of the rising amplitude in different directions is the 'arm' [b] of the 'shadow'-hypotenuse, and that is equal to 114. The square-root of the difference of the squares of that 'arm' and the 'shadow' is the 'leg', and that is equal to 2340. Then, because of the inequality of the 'arm' and the 'leg', their difference is to be added (because of the 'leg's' [being] greater) to the previously determined elapsed *ghaṭikās* of the day. Then the [new] elapsed *asus* of the day are determined [to be] equal to 5106. [Computing] with those, the Sine altitude is determined [to be] equal to 3407; the 'shadow' is equal to 461.[16] The 'arm' of the 'shadow' is equal to 285. Its 'leg' is equal to 362. Here too, because of the 'leg's' [being] greater, the difference between 'arm' and 'leg' is to be added to the elapsed *asus* of the day. The [new] elapsed *asus* of the day, thus arrived at, are equal to 5183. And when one has found with those the Sine altitude etc., [the procedure] is

[16] The calculations do not support the edition's reading of 468 for 461, so we follow the variant reading in two manuscripts of *śaśi* (one) for *vasu* (eight).

> done again [until] there is no difference from the previously found [value]. In [this] iterative rule, when the 'leg' is smaller than the 'arm', then the Sine altitude etc. is to be calculated after one has subtracted the difference of 'arm' and 'leg' from the elapsed *asus* of the day. Here, the corner altitude thus iterated becomes equal to 3414...[17]

(The omitted portions treat the computation of $\mathrm{Sin}\, a$ when the sun is on the prime vertical circle, an easy problem in similar right triangles which does not concern us here.) In this example, we are given the solar longitude Aries 15°, corresponding to a solar declination of about +6°, and the noon equinoctial shadow s_0 normalized to $R = 3438$, which is $\mathrm{Sin}\, \phi = 647$, implying a latitude of about $10;51 = 10.85°$. We then find $\mathrm{Sin}\, \eta = R \cdot \mathrm{Sin}\, \delta / \mathrm{Cos}\, \phi = 368$. Parameśvara also provides an initial guess at the time t since sunrise when the south-east *koṇaśaṅku* will occur: 8 *ghaṭikās* or 2880 *asus* or $360 \cdot 8/60 = 48$ time-degrees of the equator, represented by the equatorial arc TD in Figure 1.

To compute $\mathrm{Sin}\, a$ from this information alone, we must find (though Parameśvara does not tell us so) the hypotenuse of the right triangle in Figure 2 containing $\mathrm{Sin}\, a$ and q. This is done by exploiting the similarity of the Sine of any arc θ' measured on the sun's day-circle for a non-zero δ to that of the corresponding arc of time θ on the equator: to wit, $\mathrm{Sin}\, \theta : \mathrm{Sin}\, \theta' :: R : \mathrm{Cos}\, \delta$. We determine for the given δ and ϕ the arc of the half-equation of daylight, whose Sine OD (from similar right triangles in Figure 2) is $R \cdot \tan \delta \tan \phi = 69$, and subtract it from the time t since sunrise: $2880 - \mathrm{arcSin}\, 69 = 2811$ *asus*. (Bear in mind that δ and the quantities that depend on it in sign are positive in Parameśvara's example but negative in the figures.) The Sine of this result is the line segment $OT = 2508$, and the Sine of the corresponding arc on the day-circle is $OT \cdot \mathrm{Cos}\, \delta / R = 2494$. Our desired hypotenuse H is the sum of this amount and the Sine of the day-circle arc corresponding to OD on the equator: $2484 + \mathrm{Sin}\, \delta \cdot \mathrm{Sin}\, \phi / \mathrm{Cos}\, \phi = 2494 + 69 = 2563$.

It is then a simple matter to find $\mathrm{Sin}\, a = H \cdot \mathrm{Cos}\, \phi / R = 2517$ (slightly different, due to rounding and interpolation inaccuracies, from Parameśvara's 2516) and the 'gnomon-distance' $q = H \cdot \mathrm{Sin}\, \phi / R = 482$. Trivially, $\mathrm{Cos}\, a = \sqrt{R^2 - \mathrm{Sin}^2\, a} = 2343$, and

[17] *Goladīpikā* 4, 23 [Sarma, 49–50].

$b = q - \operatorname{Sin}\eta = 482 - 368 = 114$. Then the other leg of the triangle containing b and $\operatorname{Cos} a$ is $\sqrt{\operatorname{Cos}^2 a - b^2} = 2340$, which when diminished by b is 2226, which is added to the original time estimate in *asus*: $2880 + 2226 = 5106$. New values of all the above quantities are then calculated from the new values of t in successive iterations; the resulting sequence of approximations to $\operatorname{Sin} a$ is (2516, 3407, 3414, 3414). (Parameśvara's rule also permits one to adjust t by an arbitrarily chosen amount rather than by the difference $\sqrt{\operatorname{Cos}^2 a - b^2} - b$, but since the former technique is too ill-defined to evaluate analytically, we shall concentrate on the latter alternative.)

We can recapitulate the steps of the complete iterative procedure, given only the known ϕ and δ and an arbitrarily selected t_0, as follows:

$$\begin{aligned} H_0 &= \frac{\operatorname{Cos}\delta}{R} \cdot \operatorname{Sin}\left(t_0 - \operatorname{arcSin}\left(R \cdot \frac{\operatorname{Sin}\delta}{\operatorname{Cos}\delta}\frac{\operatorname{Sin}\phi}{\operatorname{Cos}\phi}\right)\right) \\ &\quad + \frac{\operatorname{Sin}\delta \operatorname{Sin}\phi}{\operatorname{Cos}\phi}, \\ \operatorname{Sin} a_0 &= \frac{H_0 \cdot \operatorname{Cos}\phi}{R}, \\ q_0 &= \operatorname{Sin} a_0 \cdot \frac{\operatorname{Sin}\phi}{\operatorname{Cos}\phi} = \frac{H_0 \cdot \operatorname{Sin}\phi}{R}, \\ b_0 &= q_0 - R \cdot \frac{\operatorname{Sin}\delta}{\operatorname{Cos}\phi}, \\ t_1 &= t_0^{(\text{asus})} + \sqrt{(R^2 - \operatorname{Sin}^2 a_0) - {b_0}^2} - |b_0|. \end{aligned} \tag{20}$$

In modern notation,

$$\begin{aligned} t_{n+1} = t_n^{(\text{radians})} &+ \sqrt{1 - \sin^2 a_n - \left(\sin a_n \tan\phi - \frac{\sin\delta}{\cos\phi}\right)^2} \\ &- \left|\sin a_n \tan\phi - \frac{\sin\delta}{\cos\phi}\right|. \end{aligned} \tag{21}$$

Upon setting $t_{n+1} = t_n = t$, equation (21) reduces to the quadratic of equation (8), confirming that the value of $\sin a$ at its fixed point t is indeed the desired corner altitude.

We then confront the perennial question of whether and when the prescribed iteration will actually converge to that fixed point. Rewriting it solely in terms of ϕ and δ as a new iterative function

$h(t)$, we get

$$t_{n+1} = h(t_n) = t_n \quad + \sqrt{1 - \cos^2\delta \sin^2(t_n - \sin^{-1}(\tan\delta \tan\phi)) + \frac{\sin^2\delta}{\cos^2\phi}(\sin^2\phi - 1)} \quad - \left|\sin\phi \cos\delta \sin(t_n - \sin^{-1}(\tan\delta \tan\phi)) + \sin\phi \sin\delta \tan\phi - \frac{\sin\delta}{\cos\phi}\right|. \tag{22}$$

This function is very cumbersome to evaluate analytically, but we can get an idea of its behavior by inspecting its sample orbit diagrams in Figure 7. As shown therein, the iterative function's graph is no longer a semi-ellipse but a curve oscillating about the $y = x$ line with multiple fixed points (in fact, it has an infinite number of fixed points, since the terms depending on $\sin(t_n)$ are identical for t_n, $t_n + 360°$, $t_n + 2 \cdot 360°$, etc.). However, for values of ϕ and δ for which there is a unique positive real corner altitude, there is a unique fixed point of $h(t)$ occurring in the interval from $t = 0$ to the cusp at $t = 90° + \sin^{-1}(\tan\delta \tan\phi)$—that is, in the half-day (after sunrise or before sunset) equal to 90 time-degrees plus the current half-equation of daylight. And the convergence of $h(t)$ to that fixed point is in each case sure and swift, even for latitudes as unrealistically high as 54°. Furthermore, at still higher latitudes where (for $\delta \gg 0$) there are two such fixed points, representing the occurrence of a northern corner altitude followed by a southern one in the same day, $h(t)$ will converge to the latter of these (as Parameśvara noted, addition and subtraction would have to be reversed in order to compute the former).

The extra computation required by Parameśvara's rule for the corner altitude, compared to that of Lalla, is thus rewarded by its significantly better success. In essence, while Lalla's method undertakes eventually to reconcile incompatible positions of the sun on its day-circle and on its altitude-circle, Parameśvara's computes in a more self-consistent way, keeping the sun on its own day-circle and calculating altitudes and distances that actually occur at the given day and place. Although Parameśvara nowhere discusses these issues in his explanation, it seems likely that he was inspired to experiment with *koṇaśaṅku* iterations at least in part by the desire to obtain more reliable convergence.

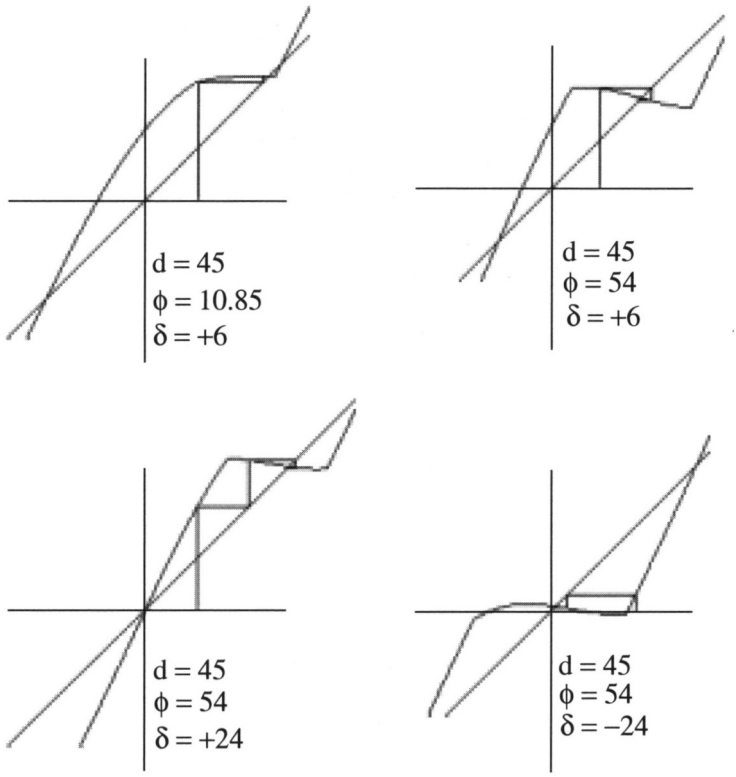

Figure 7: Orbits of $h(t)$

8 *Parameśvara's generalization of the new iterative rule*

Further evidence in support of this suggestion is provided by Parameśvara's subsequent verses, in which he combines this new iterative approach with his attempts to generalize the solution for the corner direction to any desired d:

> When one has thought of [some] desired direction [d] in a circle of [standard] radius, the Sine produced from that [angle] is determined, multiplied by the 'shadow', and divided by the radius. That is the 'arm' [or north-south distance, b] in the [given] direction. If the [actual] 'arm' of the [current] 'shadow' is equal to that, then the sun is standing in the desired direction. If it is not equal, their equality is to be established as before, from iteration.

The difference of the 'arms' is to be subtracted or added as before to the [past or future time in] *asus*, according to the [previous] explanation. And the result—plus one-half [of itself] or doubled in the case of slowness of approach to the desired amount, or minus one-third or halved if [the approach] is too fast—is always to be subtracted or added in this way in the iterative rule. [The procedure] is to be performed just like that, [and] the Sin altitude standing in the desired direction is fixed in this way by the stated rule.

[Commentary:] When one has determined the corner altitude in this way, again he states an iteration in the case of the determination of the altitude standing in [any] desired direction, [in the lines beginning] 'When one has thought of [some] desired direction...'. When the sun stands in a corner direction in a circle of [standard] radius, the Sine of one and a half [zodiacal] signs [i.e., 45°] should be the 'arm'. When [it is] in the middle [between] the directions of Agni [i.e., south-east] and Indra [east], the 'arm' is the Sine of a sign less one-quarter. When [it is] in the middle [between the directions] of Agni and Yama [south], the 'arm' is the Sine of two signs plus one-quarter. Having considered [any] desired direction in this way, and multiplied that 'arm'-Sine by the great 'shadow' [Cos a] at that time, one should divide it by the radius. Then the quotient is the 'arm' for the [desired] direction in the given circle of [radius equal to] the given 'shadow'. Again, one should compute as before [the 'arm'] of the 'shadow'-hypotenuse at that time. Then if the 'shadow-arm' is equal to the 'arm' for the desired direction, the sun is standing in the desired direction. If [it is] not equal, [their] equality should be calculated by iteration. In that case, subtraction and addition are to be done according to the [previous] explanation. One who knows the explanation is an authority on this: this is the meaning. Here, *asus* equal to the difference between the 'shadow-arm' and the 'arm' for the direction are to be added or subtracted. The approach to the quantity determined by the iterative method is not always quick. So in the case of slowness of approach, when one has increased the amount to be [e.g.,] subtracted by one-half or doubled it, and subtracted or added [that quantity], the procedure is to be done [as specified]. When, because of excessive quickness of approach, the obtained quantity exceeds the desired quantity, then [that] diminished by one-third or halved is to be taken as the amount to be added or subtracted. And in this way the altitude in [any] desired direction is determined by an iterative rule.[18]

[18] *Goladīpikā* 4, 18cd–22 [Sarma, 48].

Now Parameśvara compares b not with its 'leg' $\sqrt{\mathrm{Cos}^2\, a - b^2}$ but with the Sine of the desired direction-angle d scaled to the current Cos a, or Sin $d \cdot$ Cos a/R. Labeling this 'arm in the desired direction' b_d, we can express the steps of the iterative procedure as follows:

$$\begin{aligned}
H_0 &= \frac{\mathrm{Cos}\,\delta}{R} \cdot \mathrm{Sin}\left(t_0 - \mathrm{arcSin}\left(R \cdot \frac{\mathrm{Sin}\,\delta}{\mathrm{Cos}\,\delta}\frac{\mathrm{Sin}\,\phi}{\mathrm{Cos}\,\phi}\right)\right) \\
&\quad + \frac{\mathrm{Sin}\,\delta\,\mathrm{Sin}\,\phi}{\mathrm{Cos}\,\phi}, \\
\mathrm{Sin}\, a_0 &= \frac{H_0 \cdot \mathrm{Cos}\,\phi}{R}, \\
\mathrm{Cos}\, a_0 &= \sqrt{R^2 - \mathrm{Sin}^2\, a_0}, \\
q_0 &= \mathrm{Sin}\, a_0 \cdot \frac{\mathrm{Sin}\,\phi}{\mathrm{Cos}\,\phi} = \frac{H_0 \cdot \mathrm{Sin}\,\phi}{R}, \\
b_0 &= q_0 - R \cdot \frac{\mathrm{Sin}\,\delta}{\mathrm{Cos}\,\phi}, \\
b_{d0} &= \frac{\mathrm{Sin}\, d \cdot \mathrm{Cos}\, a_0}{R}, \\
t_1 &= t_0{}^{(\mathrm{asus})} + (b_{d0} - |b_0|).
\end{aligned} \tag{23}$$

Rewritten as a new iterative function $k(t)$, this becomes

$$\begin{aligned}
&t_{n+1} = k(t_n) = t_n + \\
&\sin d\sqrt{1 - (\cos\phi\cos\delta\sin(t_n - \sin^{-1}(\tan\delta\tan\phi)) + \sin\delta\sin\phi)^2} - \\
&\left|\sin\phi\cos\delta\sin(t_n - \sin^{-1}(\tan\delta\tan\phi)) + \sin\phi\sin\delta\tan\phi - \frac{\sin\delta}{\cos\phi}\right|.
\end{aligned} \tag{24}$$

In the formulation and discussion of this rule we see at last an explicit recognition of convergence problems.[19] Because obtaining the desired equality of b and b_d 'is not always quick', Parameśvara recommends multiplying their difference by a scale factor before

[19] Parameśvara also addressed the problem of convergence speed directly in a different work, when discussing an iterative approximation for the Sine of a given angle (see [Plofker 1996]). There too, he makes no reference to the fact that the slowly-converging iteration in question often fails to converge at all—although the new iteration he substitutes for it happens to solve both these problems.

adding it to the elapsed *asus*. The scale factor, which we shall call p, is 1.5 or 2 'in the case of slowness of approach', i.e., slow monotonic convergence; for 'excessive quickness', i.e., continually overshooting the fixed point via slow oscillating convergence, p is 2/3 or 1/2. We can rewrite this modified version of $k(t)$ as a closely related iterative function $\kappa(t)$:

$$t_{n+1} = \kappa(t_n) = t_n + p \cdot (k(t_n) - t_n). \tag{25}$$

Its behavior is illustrated by the final sample problem in verse 4, 23, quoted above: namely, calculating Sine altitude in the direction east-southeast ($d = 22.5°$). The remainder of Parameśvara's commentary on that verse explains the solution:

> [Commentary:] ... Now, for the sake of determining the Sine altitude in the middle [between the directions] of Indra and Agni, the elapsed *ghaṭikās* of the day are considered [to be] 8. [On computing] with these, the Sine altitude is determined [to be] equal to 2516. Its 'shadow' [Cos a] is equal to 2343. The 'shadow-arm' [b] should be equal to 114. Here, because the Sine altitude [is] in the middle [between the directions] of Indra and Agni, the assumed 'arm', in a circle of [standard] radius, should be [the Sine of] a fourth part of a quadrant of a circle. Its 'arm' multiplied by the 'shadow', divided by the radius, is the 'arm in the given direction'. When the arm is so much in the circle of [standard] radius, then how much [is it] in the circle [with radius] Cosine altitude? this is the proportion. The 'arm in the given direction' thus determined should be equal to 896. Here, because of the inequality of the 'arms' and the greater [size] of the 'arm in the given direction', when one has added the difference of those 'arms' to the elapsed *asus* of the day, and calculated as before the Sine altitude etc. corresponding to that elapsed [amount] of the day, the two 'arms' are to be determined [afresh]. Therefore the elapsed *asus* of the day, added to the difference of the 'arms', should be equal to 3662. The Sine altitude calculated from that is equal to 2971. Its Cosine altitude is equal to 1730. The 'shadow-arm' is equal to 201; the 'arm in the given direction' is equal to 662.
>
> Here, slowness of approach [between] the 'shadow-arm' and the 'arm in the given direction' is apparent. So when one has added their difference, doubled, to the elapsed *asus* of the day, the [quantities] corresponding to that, beginning with Sine altitude, are to be determined. The elapsed *asus* of the day, so computed, should be equal to 4584; the Sine altitude computed from that is equal

> to 3314, the Cosine altitude equal to 915. The 'shadow-arm' is equal to 266, the 'arm in the given direction' equal to 350. Their difference also, doubled, is to be added to the elapsed *asus* of the day. Then the elapsed *asus* of the day are equal to 4752, the Sine altitude equal to 3352, the Cosine altitude equal to 764. The 'shadow-arm' is equal to 274, the 'arm in the given direction' equal to 292. When one has added the difference of the 'arms', doubled in this case too, to the elapsed *asus* of the day, the resulting *asus* are equal to 4788. Their Sine altitude is equal to 3360, the Cosine altitude equal to 728. The 'shadow-arm' is equal to 276, the 'arm in the given direction' equal to 278. Now the difference of the 'arms' is only 2. The increase in the Sine altitude from the addition of that, doubled, to the elapsed *asus* of the day should be [a length corresponding to] only one arcminute. So the Sine altitude in the middle [between the directions] of Indra and Agni is equal to 3361. The Sine altitude in [any] desired direction is to be determined in the same way.[20]

Is $\kappa(t)$, the generalization of $h(t)$ to arbitrary d, an improvement upon Parameśvara's previous generalization of $g(\sin a)$? Comparing the sample results in Figure 8 with the corresponding ones in Figure 6, we observe that once again the new approach of approximating t instead of $\sin a$ is far more reliable (although in the case of Parameśvara's example, the new function actually converges a little more slowly than the old one would). So all the serious convergence problems with the original *koṇaśaṅku* iteration and its variants are at this point successfully resolved, some seven centuries after its initial appearance in Lalla's text.

9 *Conclusion*

Although fixed-point iterations play an important supporting role in much of ancient and medieval mathematical astronomy, it is very difficult to get a clear idea of how their inventors thought about them and developed them. The *koṇaśaṅku* iterations in Sanskrit texts shed some light on aspects of this question in the medieval Indian tradition. As these texts' profusion of iterative rules on similar topics suggests, there were few or no methodological qualms about the use of such approximations in place of available exact solutions. Apparently the former were not even

[20] *Goladīpikā* 4, 23 [Sarma, 51–2].

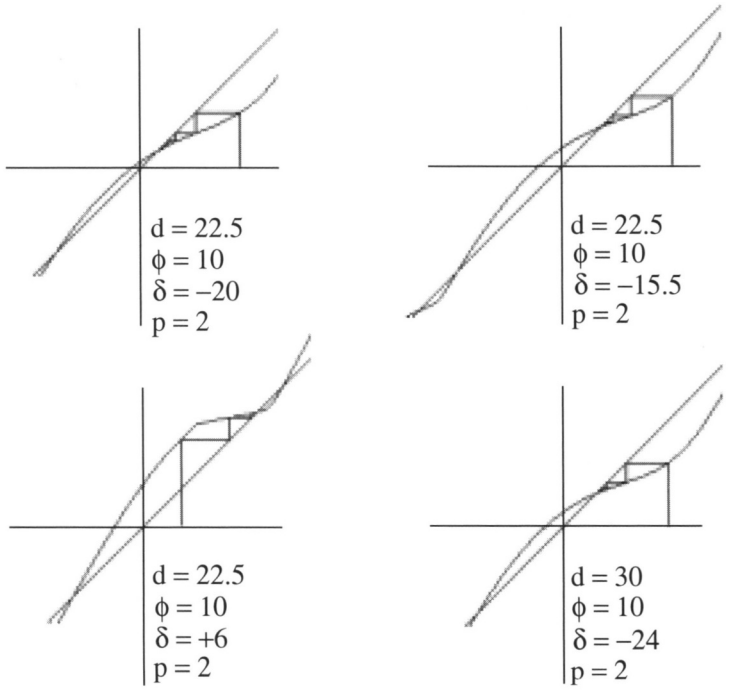

Figure 8: Orbits of $\kappa(t)$ with $d < 45°$

distinguished from the latter by terms such as 'rough' or 'approximate', as were many practical procedures like estimating π as $\sqrt{10}$ or interpolating in abbreviated Sine tables with large intervals. Evidently, it was understood that the ultimate fixed result of a (convergent) *asakṛt* or iterative rule was as accurate as the equivalent from a *sakṛt* or closed-form one.

Almost nothing is recorded about the opinions of mathematicians in this tradition concerning iterations that were not convergent. But our scrutiny of *koṇaśaṅku* rules and their results indicates that mathematicians knew a good deal more about them than met the reader's eye. The innovations that appear in the work of Mallikārjuna and Parameśvara are very satisfactorily explained as attempts to deal with the cases where Lalla's original iteration converged slowly or not at all. In Parameśvara's case, the impetus

to improve the original rule very likely came from his efforts to generalize it for arbitrary d, where its convergence failures become even more noticeable.

In fact, it is Parameśvara's work that contains the first known explicit reference to convergence problems in these rules; he also mentions in passing the existence of different approaches to the new *koṇaśaṅku* iteration, where some users 'guess' the successive approximations 'by means of one's intelligence' while 'others' rely on a deterministic algorithm. Such parenthetical remarks, in addition to the analyses discussed above, reveal glimpses of active mathematical experiment and debate among Indian mathematicians concerning the behavior of iterative rules, much richer and more complex than we might infer from their terse formulaic statements of the rules themselves.

Bibliography

B. Chatterjee, *Śiṣyadhīvṛddhidatantra of Lalla*, 2 vols, New Delhi, 1981.

R. L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Reading MA, 1992.

S. Dvivedi, *Brāhmasphuṭasiddhānta of Brahmagupta*, Benares, 1901/1902.

E. S. Kennedy & M.-Th. Debarnot, 'Al-Kāshī's Impractical Method of Determining the Solar Altitude', *Journal for the History of Arabic Science*, 3, 2, 1979, pp. 219–27.

T. S. Kuppanna Sastri, *Mahābhāskarīya of Bhāskarācārya*, Madras, 1957.

D. Pingree, *Census of the Exact Sciences in Sanskrit*, Series A, vols 1–5, Philadelphia, 1970–94.

——. 'History of Mathematical Astronomy in India', in *Dictionary of Scientific Biography*, vol. 15, New York, 1978, pp. 533–633.

K. Plofker, 'Use and transmission of iterative approximations in India and the Islamic world', in *From China to Paris: 2000*

Years Transmission of Mathematical Ideas, ed. Y. Dold-Samplonius et al., Stuttgart, 2002, pp. 167–86.

——. 'An example of the secant method of iterative approximation in a fifteenth-century Sanskrit text', *Historia Mathematica*, 23, 1996, pp. 246–56.

K. V. Sarma, *The Goladīpikā by Parameśvara*, Adyar Library Pamphlet Series, 32, Madras, 1956–7.

B. Śāstrī, *The Sūryasiddhānta*, rev. G. D. Śāstrī, Varanasi, 1989.

K. S. Shukla, *Vaṭeśvarasiddhānta and Gola of Vaṭeśvara*, 2 vols, New Delhi, 1986.