Indian Rules for the Decomposition of Fractions

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1 *Introduction*

Datta and Singh [1935. 185–203] discuss the rules for arithmetic operations with fractions in Sanskrit mathematical texts and explain the rules for the reduction of fractions to a common denominator, called *kalāsavarṇana*, meaning literally 'making [fractions] have the same color'. This reduction, common to all the Sanskrit mathematical texts available, is treated as part of the topic called *parikarman* (basic operations), and is usually classified into the following four categories:[1]

bhāgajāti (fraction class) $\frac{b}{a} \pm \frac{d}{c} = \frac{bc}{ac} \pm \frac{ad}{ac}$.

prabhāgajāti (sub-fraction class) $\frac{b}{a}\frac{d}{c} = \frac{bd}{ac}$.

bhāgānubandhajāti (partial-addition class) $a + \frac{c}{b} = \frac{ab+c}{b}$ or $\frac{b}{a} + \frac{b}{a}\frac{d}{c} = \frac{b(c+d)}{ac}$.

bhāgāpavāhajāti (partial-subtraction class) $a - \frac{c}{b} = \frac{ab-c}{b}$ or $\frac{b}{a} - \frac{b}{a}\frac{d}{c} = \frac{b(c-d)}{ac}$.

At the end of their explanation Datta and Singh discuss some rules for the decomposition of fractions in the *Gaṇitasārasaṅgraha* of Mahāvīra, and remark that 'Mahāvīra has given a number of rules for expressing any fraction as the sum of a number of unit fractions. These rules do not occur in any other work.' However, we find similar rules to decompose fractions and unity into unit fractions or ordinary fractions[2] in the *Gaṇitakaumudī* of

[1] Cf. *Brāhmasphuṭasiddhānta* 12. 2, 8–9; *Pāṭīgaṇita* 36–40; *Gaṇitasārasaṅgraha* 3. 54–137; *Mahāsiddhānta* 15. 12–13; *Siddhāntaśekhara* 13. 11–12; *Gaṇitatilaka* pp. 30–39; *Līlāvatī* 30–38; *Gaṇitakaumudī* Part 1. pp. 9–10.

[2] Indians were familiar with unit fractions. The *śulbasūtra* texts, compilations of the geometric knowledge needed for the construction of altars for Vedic sacrifice, give as an approximation of $\sqrt{2}$

$$1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

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Nārāyaṇa. In this paper I summarize the rules which Datta and Singh discussed, with examples given in the *Gaṇitasārasaṅgraha*, compare the corresponding rules in the *Gaṇitakaumudī*, and discuss the implications.

2 *Mahāvīra*

Mahāvīra wrote the *Gaṇitasārasaṅgraha* (*The Essence of Mathematics*, hereafter GSS) in about 850 A.D. and gave the rules and examples for fractions in its section dealing with the topic considered the second *vyavahāra* (practical operation) in arithmetic, namely *kalāsavarṇavyavahāra* (the operation of reduction of fractions). The text includes some numerical examples, but not the solutions to them.[3]

Mahāvīra gives the first of these rules in the *bhāgajāti* section, namely GSS *kalāsavarṇa* 55–98. This section includes the rules which Datta and Singh discussed; I summarize them as follows.
(1) To express 1 as the sum of any number (n) of unit fractions.

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-2}} + \frac{1}{2 \cdot 3^{n-2}}.$$

This rule is given in GSS *kalāsavarṇa* 75. Based on a literal translation of the versified rule,[4] the denominator of the first term is to be written as $1 \cdot 2$, and that of the last term as $3^{n-1} \cdot \frac{2}{3}$. Following the rule GSS *kalāsavarṇa* 76 gives examples when $n = 5, 6, 7$.
(2) To express 1 as the sum of an odd number of unit fractions.

$$1 = \frac{1}{2 \cdot 3 \cdot 1/2} + \frac{1}{3 \cdot 4 \cdot 1/2} + \cdots + \frac{1}{(2n-1) \cdot 2n \cdot 1/2} + \frac{1}{2n \cdot 1/2}.$$

Here $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{34}$ are expressed by ordinal numbers such as third, *tṛtīya*, fourth, *caturtha*, and thirty-fourth, *catustriṃśa*, respectively.

pramāṇaṃ tṛtīyena vardhayet taccaturthenātmacatustriṃśonena saviśeṣaḥ/

[3] The *Gaṇitasārasaṅgraha* was commented on in Kannaḍa and in Sanskrit. However, none of the commentaries has been published. See Pingree [1981. 60].

[4] *rūpāṃśakarāśīnāṃ rūpādyās triguṇitā harāḥ kramaśaḥ/*
dvidvitryaṃśābhyastāv ādimacaramau phale rūpe//

Translation

When the result is one, the denominators of the quantities having one as numerators are [the numbers] beginning with one and multiplied by three, in order. The first and the last are multiplied by two and two-thirds [respectively].

This rule is given in GSS *kalāsavarṇa* 77.
(3) To express a unit fraction $1/q$ as the sum of a number of other fractions, the numerators being given.

$$\begin{aligned}\frac{1}{q} &= \frac{a_1}{q(q+a_1)} + \frac{a_2}{(q+a_1)(q+a_1+a_2)} + \cdots + \\ &\quad \frac{a_{n-1}}{(q+a_1+a_2+\cdots+a_{n-2})(q+a_1+a_2+\cdots+a_{n-1})} + \\ &\quad \frac{a_n}{a_n(q+a_1+a_2+\cdots+a_{n-1})},\end{aligned}$$

where $a_1, a_2, \cdots, a_n$ and q are given. This is given in GSS *kalāsavarṇa* 78. GSS *kalāsavarṇa* 79 provides examples where $a_1 = 7, a_2 = 9, a_3 = 3, a_4 = 13$ for $q = 1$, 4, and 6.
(4) To express any fraction p/q as the sum of unit fractions.
Let the number i be so chosen that $\frac{q+i}{p}$ is an integer r; then

$$\frac{p}{q} = \frac{1}{r} + \frac{i}{r \cdot q}.$$

For the second term the same procedure is performed. This is given in GSS *kalāsavarṇa* 80. Gupta [1993] explains this rule with some examples. GSS 81 contains an example requiring the denominators of three unit fractions the sum of which is $\frac{1}{4}$, and those of four unit fractions whose sum is $\frac{3}{7}$.
(5) To express a unit fraction as the sum of two other unit fractions.

$$\frac{1}{n} = \frac{1}{p \cdot n} + \frac{1}{\frac{p \cdot n}{p-1}}.$$

$$\frac{1}{a \cdot b} = \frac{1}{a(a+b)} + \frac{1}{b(a+b)}.$$

This is given in GSS *kalāsavarṇa* 85. GSS *kalāsavarṇa* 86 is an example thereof where n or $a \cdot b$ equals 6 or 10.
(6) To express any fraction as the sum of two other fractions whose numerators are given.

$$\frac{p}{q} = \frac{a}{\frac{ai+b}{p} \cdot \frac{q}{i}} + \frac{b}{\frac{ai+b}{p} \cdot \frac{q}{i} \cdot i}$$

when p, q, a, b are given and i, such that $ai + b$ is to be divided by p without remainder, is to be found. This is stated in GSS *kalāsavarṇa* 87. The example in GSS *kalāsavarṇa* 88 seeks the denominators of the two unit fractions whose sum is $\frac{1}{4}$, and also those of the two fractions whose numerators are 7 and 9 respectively and whose sum is $\frac{2}{5}$.

Pṛthūdaka (*fl.* 864), a contemporary of Mahāvīra, in his commentary on 'the first *jāti*', that is the *bhāgajāti*, in the *Brāhmasphuṭasiddhānta* (12. 8) written by Brahmagupta in 628, offers an example[5] requesting the sum of

$$\frac{5}{22}+\frac{7}{66}+\frac{9}{38}+\frac{1}{39}+\frac{4}{13}+\frac{11}{114}.$$

When these quantities are rearranged in the order

$$\frac{5}{22}+\frac{7}{66}+\frac{9}{38}+\frac{11}{114}+\frac{4}{13}+\frac{1}{39},$$

there occur three consecutive pairs. The first pair (with 22 and 66 as denominators) can be produced from GSS Rule 6 with $p = 1, q = 3, a = 5, b = 7$ and $i = 3$. The second pair is also obtainable from the same rule when $p = 1, q = 3, a = 9, b = 11$ and $i = 3$. The third and last pair results when $p = 1, q = 3, a = 4, b = 1, i = 3$. Pṛthūdaka might have known this rule and used it to construct his sample problem.

(7) Datta and Singh mention a particular case of Rule 6, described in GSS *kalāsavarṇa* 89:

$$\frac{p}{q}=\frac{a}{\frac{aq+b}{p}}+\frac{b}{\frac{aq+b}{p}\cdot q},$$

when p, q, a, b are given, provided that $(aq + b)$ is divisible by p. Sample problems in GSS *kalāsavarṇa* 90–92 require the denominators of the two unit fractions whose sum is $\frac{2}{7}$; the denominators of fractions which have 6 and 8 respectively as numerators and whose sum is also $\frac{2}{7}$; the two unit fractions that sum to $\frac{1}{15}$ when

[5] *dviyamā rasaṣaṭkāś ca vasulokā navāgnayaḥ/*
trīndavaḥ kṛtarudrāś ca chedasthāne prakalpitāḥ//
pañcāgā nava rūpaṃ ca vedā rudrās tadaṃśakāḥ/
militā yatra dṛśyante kas tatra dhanasañcayaḥ//
Quoted by Dvivedī in his edition of the *Brāhmasphuṭasiddhānta* p. 176. This stanza is found in folio 48a.

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}$ is given; and the two fractions whose sum is $\frac{1}{20}$ and whose denominators are 7 and 11 respectively, when $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$ is given.

Toward the end of their discussion Datta and Singh mention a rule 'to express a given fraction as the sum of an even number of fractions whose numberators are previously assigned'. This is their translation:[6] 'After splitting up the sum into as many parts, having one for each of their numerators, as there are pairs (among the given numerators), these parts are taken as the sum of the pairs, and (then) the denominators are found according to the rule for finding two fractions equal to a given unit fraction.'

We add two remarks not mentioned by Datta and Singh.

1) The examples to find the sum of the given fractions in GSS *kalāsavarṇa* 60–62 can be expressed as follows:

$$\frac{a}{a(a+1)} + \frac{a}{(a+1)(a+2)} + \frac{a}{(a+2)(a+3)} + \cdots + \frac{a}{\{(a+(n-1)\}(a+n)} + \frac{a}{a+n}.$$

2) The examples in GSS *kalāsavarṇa* 83–84 require the denominators of three unit fractions whose sum is $\frac{23}{12}$, and the three denominators whose numerators are given as 3, 7 and 9 when the sum of those three fractions is $\frac{73}{46}$. As the solution for these problems GSS *kalāsavarṇa* 82 gives the following rule:

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} = \frac{p}{q},$$

where $a_1, a_2, \cdots, a_n$ and p, q are given. One has to find $i_1, i_2, \cdots, i_n$ such that $a_1 i_1 + a_2 i_2 + \cdots + a_n i_n = p$, then $b_n = q/i_n$.

3 *Nārāyaṇa Paṇḍita*

Nārāyaṇa Paṇḍita wrote the *Gaṇitakaumudī (Moonlight of Mathematics)* in 1356.[7] The *Gaṇitakaumudī* (hereafter GK) consists of the *mūla* (root or original), that is, versified rules (*sūtra*) and examples (*udāharaṇa*), and of a prose commentary (*vāsanā*) thereon. The answers to the worked examples are given in the

[6] In the footnote they identify this stanza as GSS 89, but the correct stanza number is 93.

[7] For the author Nārāyaṇa Paṇḍita, see Kusuba [1994. 1–5].

vāsanā. Nārāyaṇa gave the four simple rules for reduction of fractions (discussed above in the Introduction) in the *parikarman.* However, he devoted the twelfth chapter, named *aṃśāvatāra-vyavahāra* (the operation of the appearance of fractions) to additional rules for fractions. The eight rules in the section called *bhāgajāti* in that chapter are of five sorts:

1) to decompose 1 to a sum of unit fractions (Rules 1–2)
to decompose a given fraction to the sum of unit fractions (Rule 3)

2) to decompose 1 to a sum of arbitrary fractions (Rule 4)

3) to decompose 1 to the sum of fractions whose numerators are given (Rules 5–6)

4) to find denominators of fractions with given numerators, summing to a given result (Rule 7)

5) to find numerators when denominators and the result (sum) are given (Rule 8)

I give a critical edition of the rules with the English translation and explain the rules and some of examples thereof.[8]

Rule 1

ekādyekacayānāṃ dvayor dvayor nikaṭayor vadhāś chedāḥ/
yo 'ntyaḥ so 'ntyaharaḥ syād yoge rūpaṃ tad iṣṭaphalaguṇitam//1//

The products of two successive [numbers] beginning with one and increasing by one are denominators. The one which is last is the last divisor (i.e., denominator). When [they are] added together, [the result is] one. That is multiplied by any result.

$$1 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n}.$$

Example thereof

The number of terms is given as 6. This is set out thus in the *vāsanā.*

1	1	1	1	1	1	phalam 1/
0	0	0	0	0	0	

[8] I have used three manuscripts. For their sigla, see Bibliography.

Here the zero signs indicate unknown numbers which are required. The word *phalam*, which literally means fruit, indicates the result. The answer given is

$$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{6}.$$ [9]

Nārāyaṇa's rule seems to be more general than GSS Rule 2, a similar rule for an odd number of terms. When in GSS Rule 3 $a_1 = a_2 = \cdots = a_n = 1$ and $q = 1$, it reduces to GK Rule 1. Pṛthūdaka, whom we mentioned above, might have known GK Rule 1 or a similar application of GSS Rule 3, because in his commentary he includes an example requiring the sum of

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{4}.$$ [10]

Rule 2

ekāditriguṇottaravṛddhyāṅkasthānasammitāś chedāḥ/
ādyantau ca dviguṇāv antyas trihato 'ṃśake rūpam//2//
2c ca] vada; 2d aṃśaka NRV

When there is unity in numerator the denominators are measured by the [number of] places of the numbers beginning with one and increasing by [their] triples. The first and the last are multiplied by two. The last is multiplied by three.

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-2}} + \frac{1}{2 \cdot 3^{n-2}}.$$

Rule 2 is an alternative rule for the decomposition of 1 to the sum of unit fractions. The word *trihato* (multiplied by three) should be emended to *trihṛto* (divided by three); otherwise the denominator of the last term becomes $2 \cdot 3^n$.

[9] The *vāsanā* gives the answer to another problem whose solution is half that of the previous one:

1	1	1	1	1	1
4	12	24	40	60	12

[10] *rūpārdhaṃ rūpaṣaḍbhāgo rūpāṃśo dvādaśas tathā/*
rūpasya ca turīyāṃśaḥ ko 'rthaḥ sampiṇḍite bhavet//
Quoted by Dvivedī p. 176. This stanza is found on folio 48a.

There is no example for Rule 2, but the *vāsanā* gives an answer to the case where the number of the terms is 6 as in the previous example.

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{162}.$$[11]

Rule 3

phalahāro 'bhīṣṭayutaḥ phalāṃśabhakto yathā bhavec chuddhiḥ/
labdhiś chedo bhāgaṃ phalataḥ saṃśodhayec ca tacchesam//3//
tasmād utpādyānyaṃ śeṣam upāntyāṅkaśeṣaṃ ca/
ekaikeṣv aṃśeṣu kramo 'yam āryoditaḥ spaṣṭaḥ//4//
3b -bhaktau NV, 3c labdhi- N, chedo] kṣepo NRV, 3d read yaccheṣam, 3] numbered 5 R, not numbered NV, 4ab om.V, 4a utpādyāntyaṃ N, 4b upāntyakāśeṣam N, upāntyakaḥ śeṣam R, ca om.NR, 4d yam] cam NV, tvam R, āryoditaspaṣṭā NRV, numbered 5 R, not numbered NV

[One should suppose an arbitrary number] such that the divisor of the result added to an arbitrary number and [then] divided by the numerator of the result leaves no remainder. The quotient is the denominator. One should subtract the fraction from the result. Having produced from what remains another remainder and the remainder from the penultimate number, [one should operate in the same way] for each fraction. This procedure which was told by the noble man is evident.

A fraction p/q, which is the "result," is given, and one is to find denominators of unit fractions that sum to the "result." An "arbitrary number" i is to be determined so that the quotient $(q+i)/p$ is an integer; this quotient is the first desired denominator, and the numerator is always 1. Therefore the next "result" is $\frac{p}{q} - \frac{1}{\frac{q+i}{p}}$, and the next denominator is found in the same way by assuming a new i.

An example is given in the *vāsanā* for an alternative solution to the previous problem, in which the result is equal to 1, that is 1/1, and the number of terms is six. In this case the *vāsanā* gives

[11] Here also the *vāsanā* gives a solution to another problem which halves the previous one:

1	1	1	1	1	1
4	6	18	54	162	324

the numbers 1, 1, 1, 1, 1 as the arbitrary numbers. The operation is as follows:

$\frac{1+1}{1} = 2, 1 - \frac{1}{2} = \frac{1}{2}$; $\frac{2+1}{1} = 3, \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$;
$\frac{6+1}{1} = 7, \frac{1}{6} - \frac{1}{7} = \frac{1}{42}$; $\frac{42+1}{1} = 43, \frac{1}{42} - \frac{1}{43} = \frac{1}{1806}$;
$\frac{1806+1}{1} = 1807$; $\frac{1}{1806} - \frac{1}{1807} = \frac{1}{3263442}$.
Therefore

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \frac{1}{3263442}.$$

Two problems with multiple solutions are given. In both cases the number of terms is four. In the first case the given fraction is 5/6.

$\frac{5}{6} = \frac{1}{2} + \frac{1}{4} + \frac{1}{13} + \frac{1}{156}$ (the arbitrary numbers are 4,1,1 in order.)
$= \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{120}$ (the arbitrary numbers are 4,2,1 in order.)[12]
$= \frac{1}{3} + \frac{1}{5} + \frac{1}{4} + \frac{1}{20}$ (the arbitrary numbers are 9,3,2 in order.)

In the second case, on the other hand, the given fraction is 7/9.
$\frac{7}{9} = \frac{1}{2} + \frac{1}{4} + \frac{1}{37} + \frac{1}{1332}$ (the arbitrary numbers are 5,2,1 in order);
$= \frac{1}{4} + \frac{1}{2} + \frac{1}{38} + \frac{1}{684}$ (the arbitrary numbers are 19,2,2 in order.).
According to the Egyptian-Greek method discussed by Knorr [1982], each of these fractions may be written in one and only one way: $\frac{5}{6} = \frac{2+3}{6} = \frac{2}{6} + \frac{3}{6} = \frac{1}{3} + \frac{1}{2}$; $\frac{7}{9} = \frac{1}{3} + \frac{1}{4} + \frac{1}{9} + \frac{1}{12}$. On the other hand the GK does not yield unique solutions, but rather allows many answers according to the consistent use of a particular computational procedure. After stating the answers the *vāsanā* reads: *evam iṣṭavaśād bahudhā* (Thus there are many ways according to the [choice of] arbitrary [numbers].)

The procedure in GK Rule 3 is equivalent to GSS Rule 4; after stating it, the GK comments '*kramo 'yam āryoditaḥ spaṣṭaḥ* (this procedure which was told by the noble man is evident).' It is not certain whether Nārāyaṇa is referring to Mahāvīra or someone else.

Rule 4

parikalpyeṣṭān aṅkān ādyaḥ kandābhidho 'ntimo 'grākhyaḥ/
nijapūrvaghno hi paro 'ntaraṃ harāṃśau kramāt syātām//5//

[12] The last two numbers, 2,1, are missing in the edition and all the manuscripts.

antyāgracchedaḥ syād rūpaṃ cāṃśo 'tha te 'ṃśakāḥ sarve/
kandavinighnās teṣāṃ saṃyogo jāyate rūpam//6//
5c ntaraṃ om. NRV, 5 numbered 6 NRV, 6a read antye 'graś chedaḥ, 6b cāṃśo 'tha] cāṃśātha NRV, 6d saṃyogo P, 6 numbered 7 NRV

Supposing arbitrary numbers, [one] calls the first [number] *kanda*, and the last *agra*. [Each] one multiplied by its previous one, and the difference [between them], are the divisor and the numerator, in order. For the last [term] the denominator is the *agra* and the numerator is unity. All these numerators are multiplied by the *kanda*. Their sum is unity.

The arbitrary numbers are $k_1, k_2, k_3, k_4, \cdots, k_n$, where k_1 is called *kanda* (root), and k_n *agra* (tip).

$$1 = \frac{(k_2 - k_1)k_1}{k_2 \cdot k_1} + \frac{(k_3 - k_2)k_1}{k_3 \cdot k_2} + \cdots + \frac{(k_n - k_{n-1})k_1}{k_n \cdot k_{n-1}} + \frac{1 \cdot k_1}{k_n}.$$

Example thereof

The number of terms is equal to 6; the successive k_i are $1, 2, 3, 4, 5, 6$ in order.

$$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{6}.$$

In this case the result is same as what was derived from Rule 1 (see example above). Another example given in the *vāsanā* is:

$$1 = \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{2}{7}\,(k_i = 2, 3, 4, 5, 6, 7).$$

The *vāsanā* enumerates the following results.

$$1 = \frac{1}{4} + \frac{3}{20} + \frac{1}{10} + \frac{1}{14} + \frac{3}{56} + \frac{3}{8}\,(k_i = 3, 4, 5, 6, 7, 8).$$

$$1 = \frac{2}{3} + \frac{2}{15} + \frac{2}{35} + \frac{2}{63} + \frac{2}{99} + \frac{1}{11}\,(k_i = 1, 3, 5, 7, 9, 11).$$ [13]

$$1 = \frac{2}{3} + \frac{5}{24} - \frac{3}{40} - \frac{3}{10} + \frac{5}{14} + \frac{1}{7}\,(k_i = 1, 3, 8, 5, 2, 7).$$ [14]

[13] The published edition reads 2/11 for 1/11.

[14] The published edition and the manuscripts read

$$1 = \frac{2}{3} + \frac{5}{24} - \frac{3}{40} - \frac{3}{10} + \frac{51}{147}.$$

And the edition reads $k_i = 1, 3, 8, 5, 2, \frac{98}{83}$.

Negative numbers are usually indicated by a dot placed above them.

Rule 5

> *parikalpyādau rūpaṃ sāṃśaṃ parataḥ paraṃ tad eva syāt/*
> *nikaṭavadhas tacchedāḥ prāntyo yo 'ṅkaḥ sa eva tacchedaḥ//7//*
> sāṃśaḥ NRV, 7d tacchedāḥ NRV, numbered 18 NRV.
>
> Assuming unity first [one should] add to the [given] numerators successively. The product of [two] successive [added numbers] [gives] their denominators. The number which is last is itself its denominator.

This is a case where the numerators of fractions summing to 1 are given. If these numerators are indicated by a_i, and it is required to calculate $i_1, i_2, i_3, i_4, \cdots, i_n$ such that
$i_1 = a_1 + 1$,
$i_2 = a_2 + i_1$, which is equal to $a_2 + (a_1 + 1)$,
$i_3 = a_3 + i_2$,
$i_4 = a_4 + i_3 \cdots$
This is equivalent to the following formula:

$$1 = \frac{a_1}{1 \cdot i_1} + \frac{a_2}{i_1 \cdot i_2} + \frac{a_3}{i_2 \cdot i_3} + \cdots + \frac{a_n}{i_{n-1} \cdot i_n} + \frac{1}{i_n}.$$

The *sūtra* does not explicitly state that the last numerator is 1. This rule can be derived from GSS Rule 3 when $q = 1$. Example thereof in the GK gives a problem in which the numerators are the integers beginning with 3 and increasing by 2 in four places.[15] The setting for this problem is as follows:

3	5	7	9	1	phalam 1/
0	0	0	0	0	

Answer
$1 = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} + \frac{1}{25}$ because $i_1 = 4, i_2 = 9, i_3 = 16$ and $i_4 = 25$. The *vāsanā* gives five denominators in the answer to the problem although the number of places is given as four.

[15] *aṃśās trikādidvicayāś caturṣu sthāneṣu tacchedanakāś ca kaiścit/ saṃyojitā yena lavena rūpaṃ bhaved dhi tatrātha harān vadāśu//*

Rule 6

> *utpādayoś ca bhāgān yugmamite tadyutau yathā rūpam/*
> *tacchedahatoddiṣṭāṃśakaḥ parāṃśādhikas tu pūrvaharaḥ//8//*
> *so 'pi haraghnas tu paro hara evaṃ nikhilayugmeṣu/*
> *viṣamapadeṣu tathā prāntyaharaghnoddiṣṭabhāgaś ca//9//*
> *chedaḥ syād antyastho nijayugmalavair hṛtāś chedāḥ/*
> 8a read utpādayec 8b yugmamites NRV, 8] numbered 9 NRV, 9b evaṃ] evā NRV, 9d -bhāgaghnaḥ (ca om.) NRV, 9] numbered 10 NRV, 10a antyascho R, 10b -lavo hṛtau NRV.
>
> When [the numbers of the fractions] are taken in pairs, one should produce fractions in such a way that their sum is unity. The indicated numerator multiplied by that denominator and increased by the other numerator is the first divisor. That multiplied by the divisor is the other divisor. [One should operate] thus for all the pairs. For an odd [number of] terms [one should operate] thus, [but] the indicated numerator multiplied by the last divisor is the denominator placed last. The denominators are divided by the numerators of [the fractions for] their own pairs.

This is another case where the numerators $a_1, a_2, \cdots, a_n$ of the fractions whose sum is equal to one are given and one has to find their denominators.

When n is an even number, let $n = 2m$. First, from the previous rule one must find $b_1, b_2, \cdots, b_m$ such that

$$1 = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} + \cdots + \frac{1}{b_m}.$$

Then the GK gives the following formula for expressing each of the $1/b_i$ in terms of two of the given numerators:

$$\frac{1}{b_i} = \frac{a_{2i-1}}{a_{2i-1} \cdot b_i + a_{2i}} + \frac{a_{2i}}{b_i(a_{2i-1} \cdot b_i + a_{2i})}.$$

This rule can be deduced from GSS Rule 6 when $p = 1, q = b_i, a = a_{2i-1}, b = a_{2i}$.

When n is an odd number, on the other hand, let $n = 2m - 1$. Then

$$1 = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} + \cdots + \frac{1}{b_{m-1}} + \frac{1}{b_m}.$$

For the first $m - 1$ terms one proceeds as above; the last term is rewritten as

$$\frac{1}{b_m} = \frac{a_n}{a_n \cdot b_m}.$$

If the numerator of any b_i is not unity, one has to divide each of its denominators by that numerator.
Example thereof

Six numerators 3,5,7,9,11,13 and the result 1 are given. The *vāsanā* runs:

Text

nyāsaḥ

3	5	7	9	11	13
0	0	0	0	0	0

phalam 1/ ṣaṭsu padeṣu yugatrayaṃ vartate/ yugamite rūpotpannabhāgāḥ

1	1	1
2	6	3

Translation

The setting.

3	5	7	9	11	13
0	0	0	0	0	0

The result is 1. In six terms there occur three pairs. The fractions produced from one in [terms] taken in pairs are

1	1	1
2	6	3

.

This shows that 1 is divided into the sum of three fractions such as $\frac{1}{2}, \frac{1}{6}, \frac{1}{3}$. Then for the first fraction, $\frac{1}{2}$, one applies the rule with the numerators 3,5.

$$\frac{1}{2} = \frac{3}{11} + \frac{5}{22}.$$

In a similar way one finds: $\frac{1}{6} = \frac{7}{51} + \frac{9}{306}$ and $\frac{1}{3} = \frac{11}{46} + \frac{13}{138}$. Therefore

$$1 = \frac{3}{11} + \frac{5}{22} + \frac{7}{51} + \frac{9}{306} + \frac{11}{46} + \frac{13}{138}.$$

Alternative solution: from the equation $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, the rule produces another answer as follows:

$$1 = \frac{3}{14} + \frac{5}{42} + \frac{7}{30} + \frac{9}{90} + \frac{11}{46} + \frac{13}{138}.$$ [16]

In the former case $\frac{9}{306}$ can be reduced to $\frac{1}{34}$. In the latter case also $\frac{9}{90}$ can be reduced to $\frac{1}{10}$. However, the *vāsanā* says as follows

[16] The published edition reads 1/10 instead of 9/90.

at the end of the next solution: "When the reduction of the indicated numerator and the denominator [obtained] is made there would occur change of the indicated numerators. Therefore the reduction is not to be made."[17]

The *vāsanā* gives another alternative solution illustrating the technique for dealing with non-unity numerators, by using Rule 4 with the arbitrary numbers $k_i = 1$, 3, and 5 to obtain the equation $1 = \frac{2}{3} + \frac{2}{15} + \frac{1}{5}$. Then Rule 6 is employed to produce $\frac{1}{3} = \frac{3}{14} + \frac{5}{42}, \frac{1}{15} = \frac{7}{114} + \frac{9}{1710}, \frac{1}{5} = \frac{11}{68} + \frac{13}{340}$.

Because the numerator for the denominator 3 is not unity we have to divide both its derived denominators, 14 and 42, by its numerator, 2. Therefore $\frac{2}{3} = \frac{3}{7} + \frac{5}{21}$. In the same way $\frac{2}{15} = \frac{7}{57} + \frac{9}{855}$ is obtained. Hence

$$1 = \frac{3}{7} + \frac{5}{21} + \frac{7}{57} + \frac{9}{855} + \frac{11}{68} + \frac{13}{340}.$$

The *vāsanā* gives the following answer. From the equation $1 = \frac{4}{5} + \frac{4}{45} + \frac{1}{9}$[18]

$$1 = \frac{3}{5} + \frac{5}{25} + \frac{7}{81} + \frac{9}{3645} + \frac{11}{112} + \frac{13}{1008}.$$

When the number of numerators is odd, such as when the numerators are $3, 5, 7, 9, 11, 13, 15$, one has an expression consisting of four terms such as $1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{4}$, which might be obtained by means of Rule 1. According to the procedure in Rule 6 for odd n,

$$1 = \frac{3}{11} + \frac{5}{22} + \frac{7}{51} + \frac{9}{306} + \frac{11}{145} + \frac{13}{1740} + \frac{15}{60}.$$

Other answers are given:

$$1 = \frac{3}{17} + \frac{5}{68} + \frac{7}{37} + \frac{9}{148} + \frac{11}{57} + \frac{13}{228} + \frac{15}{60}$$ [19]

is derived from $1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$;

$$1 = \frac{3}{7} + \frac{5}{21} + \frac{7}{57} + \frac{9}{855} + \frac{11}{199} + \frac{13}{6965} + \frac{15}{105}$$

[17] *yady uddiṣṭāṃśachedayor apavartane kṛte taduddiṣṭāṃśānāṃ vikṛtir bhavati yato 'pavartanaṃ na kartavyaṃ/* taduddiṣṭāṃśānāṃ] uddiṣṭānāṃ; yato] tadā tayor in the published edition.

[18] $\frac{1}{9}$ is omitted in both the printed edition and the manuscripts.

[19] The term 7/37 is missing in both the printed edition and the manuscripts.

is derived from $1 = \frac{2}{3} + \frac{2}{15} + \frac{2}{35} + \frac{1}{7}$ obtained by Rule 4 with the arbitrary numbers $1, 3, 5, 7$.

Rule 7

uddiṣṭāṃśe prathame phalahāraghne parāṃśasaṃyukte/
phalabhāgāpte vyagre hāraḥ syāt phalaharaghno 'ntyaḥ//10//
śuddhir na bhaved yadi vālpo 'ṃśo bhājyaṃ tathetaraḥ kṣepam/
hāraḥ phalāṃśa iti vā kuṭṭakena sakṣepakā labdhiḥ//11//
chedaḥ syāt phalahārād alpo 'nalpaḥ phalacchedam/
kramaśo vibhajed guṇayed yatra na śuddhis tad eva khilam//12//
10 numbered 11 NRV, 11c iti vā om. NRV, 11 numbered 12 NRV, 12c guṇayed vibhajed NRV, 12 numbered 13 NRV.

When the former numerator indicated is multiplied by the divisor of the result, and added to the other numerator, and divided by the numerator of the result without any remaninder, [the quotient is] the divisor. [That quotient] multiplied by the divisor of the result is the latter [denominator]. When it is not divisible, the quotient with the addendum (i.e., the general solution) [is obtained] by means of the indeterminate equation such that the smaller numerator is the dividend, the other [numerator] is the addendum, and the numerator of the result is the divisor. According to whether the denominator [obtained] is smaller or greater than the divisor of the result, one should divide or multiply the denominator of the result respectively. If it is not divisible it is insoluble.

When two numerators a_1 and a_2 and the result a/b are given, then the corresponding denominators x and y are to be found such that $\frac{a_1}{x} + \frac{a_2}{y} = \frac{a}{b}$, as follows:

$$x = \frac{a_1 b + a_2}{a}, y = bx.$$

This rule, which implies that x is to be an integer, is equal to GSS Rule 7.

If $a_1 b + a_2$ is not divisible by a, Nārāyaṇa uses an indeterminate equation, which he discusses in chapter 10 of the GK.

Example

when $\frac{1}{x} + \frac{1}{y} = \frac{1}{20}$, $x = \frac{1\times 20+1}{1} = 21, y = 20 \times 21 = 420$;

when $\frac{3}{x} + \frac{7}{y} = \frac{2}{5}$, $x = \frac{3\times 5+7}{2} = 11, y = 11 \times 5 = 55$;

when $\frac{3}{x} + \frac{5}{y} = \frac{71}{70}$, $x = \frac{3\times 70+5}{71} = \frac{215}{71}$, which is not an integer. In this case one has to find x by means of an indeterminate

equation: According to the rule, the smaller numerator (3) is the dividend, the greater numerator (5) the addendum, and the numerator of the result (71) is the divisor. Therefore in this case it is required to find some integers x and z satisfying
$x = \frac{3z+5}{71}$.

When z is equal to any positive integer t multiplied by 71 and increased by 22, x will be an integer of the form $3t+1$; so for $t = 0, 1, 2, 3, \cdots$ we find $x = 1, 4, 7, 10, \cdots$. No such solution for z appears in the *vāsanā*, but the value of x is chosen to be 10. Again according to the rule, since this x is smaller than the "divisor of the result" b, then $y = b/x$ or $70/10 = 7$. Therefore $\frac{3}{10} + \frac{5}{7} = \frac{71}{70}$.

Rule 8

The last rule of the *bhāgajāti* is for a case where denominators are given and the numerators are to be found.

> *ajñāteṣv aṃśeṣu prakalpya rūpaṃ pṛthak pṛthak cāṃśān/*
> *kṛtvā tulyacchedān phalahāreṇa cchido lopyāḥ//13//*
> *teṣu dvayoḥ kayościd hāras tv ekaḥ paraś ca ṛṇabhājyaḥ/*
> *iṣṭāṃśahatānyonitaphalaṃ bhavet kṣepako 'tha dṛḍhakuṭṭāt//14//*
> *guṇalabdhī sakṣepe vibhājyaharayor lavau syātām/*
> *harabhājyakṣepāṇāṃ yathāpavartas tathāṃśakā kalpyāḥ//15//*
> 13 numbered 14 NRV, 14a dvayor dvayościd NV, 14b ṛṇabhājyaḥ] bhājyor ṇam NRV, 14c -hato nyonita- NRV, 14d kuṭṭān V, 14 numbered 15 NRV, 15a -labdhi V, 15c hara- V, 15 numbered 16 NRV.
>
> When the numerators are unknown, assuming each numerator to be unity and making the denominators equal to divisors of the result, one should remove the denominators [newly obtained]. One of a certain pair among them is the divisor, and the other the negative dividend. The result decreased by the assumed numerators multiplied by the other [numerators] is the addendum. Then the multiplier and the quotient accompanied by the addendum (i.e, general solutions) [obtained] from the fixed indeterminate equation will be numerators for [the two denominators chosen as] the dividend and the divisor. The numerators should be assumed in such a way that reduction of the divisor, the dividend, and the addendum is possible.

I will explain this rule by means of an example given as Example 9. The *vāsanā* writes

0	0	0	0	phalam	3
5	8	9	12		1
					40

One has to find the numerators of four fractions whose denominators are 5, 8, 9, and 12 respectively. The sum of these fractions is $3\frac{1}{40}$.[20]

First one has to suppose all the numerators to be 1 and reduce them to the same denominator; $\frac{1}{5} = \frac{72}{360}, \frac{1}{8} = \frac{45}{360}, \frac{1}{9} = \frac{40}{360}, \frac{1}{12} = \frac{30}{360}$. $3\frac{1}{40}$ is equal to $\frac{1089}{360}$. Then one supposes two arbitrary numbers as the numerators for two of them. The *vāsanā* supposes 2 as the numerator of the denominator 5, and 1 as the numerator of the denominator 8. Then one has to find corresponding numerators for the two remaining denominators 9, 12. The *vāsanā* first calculates the combined value of the remaining numerators: $1089 - (72 \times 2 + 45 \times 1) = 900$.[21] Suppose this to be the addendum, one of the numerators with the denominator 360 to be the negative dividend, and the other numerator to be the positive divisor of the indeterminate equation that satisfies $900/360 = 30y/360 + 40x/360$:

$$\frac{-40x+900}{30} = y, \text{ which is reduced to } \frac{-4x+90}{3} = y.$$

The general solution is $y = -4t + 30, x = 3t$. The list of answers given is:

[20] Read khābdhy- for svābdhy- in *udāharaṇa* 9b.

[21] In the footnote on this operation, the printed text reads:

> *atra truṭir asti pustakadvaye 'pi* (p.272)
>
> (Here there occurs an omission in both the manuscripts.)

In the preface to the printed text, Padmākara Dvivedī says that he discovered a manuscript of the *Gaṇitakaumudī* in the collection of his father, Sudhākara, after the father's death. Padmākara published his edition based on this manuscript only. In the footnote at page 286, Padmākara says again, '*atrobhayatra truṭiḥ/* (here there occurs an omission in both.)' In Part 1 among the two volume published edition he refers to other readings (*pāṭhāntara*) at pages 25, 27, 32, 33, 44, 49, 53. At the Sanskrit introduction and at footnote at page 165 of the second volume he mentions the Nepal manuscript (*nepālaprāptapusta*) and says that its reading is similar to (*pratilipisamāna*) that of the manuscript in his hand.

2	1	3	26
2	1	6	22
2	1	9	18
2	1	12	14
2	1	15	10
2	1	18	6
2	1	21	2

This means: when one takes the case $t = 1$, the answer to the problem is $\frac{2}{5} + \frac{1}{8} + \frac{3}{9} + \frac{26}{12} = 3\frac{1}{40}$. In the same way we can obtain equivalent solutions for successive values of t: $3\frac{1}{40} = \frac{2}{5}+\frac{1}{8}+\frac{6}{9}+\frac{22}{12} = \frac{2}{5}+\frac{1}{8}+\frac{9}{9}+\frac{18}{12} = \frac{2}{5}+\frac{1}{8}+\frac{12}{9}+\frac{14}{12} = \frac{2}{5}+\frac{1}{8}+\frac{15}{9}+\frac{10}{12} = \frac{2}{5}+\frac{1}{8}+\frac{18}{9}+\frac{6}{12} = \frac{2}{5}+\frac{1}{8}+\frac{21}{9}+\frac{2}{12}$. In this way the *vāsanā* merely enumerates a great number of possible solutions, some of which are reproduced below:

2	3	3	23
2	3	6	19
2	3	9	15
2	3	12	11
2	3	15	7
2	3	18	3

2	5	3	20
2	5	6	16
2	5	9	12
2	5	12	8
2	5	15	4

2	7	3	17
2	7	6	13
2	7	9	9
2	7	12	5
2	7	15	1

2	9	3	14
2	9	6	10
2	9	9	6
2	9	12	2

2	11	3	11
2	11	6	7
2	11	9	3

4 *Conclusions*

This survey attests to a remarkable continuity of computational tradition from Mahāvīra to Nārāyaṇa despite the five centuries for which we know of no representatives of that tradition. Some of Nārāyaṇa's rules are equivalent to or can be deduced from Mahāvīra's as the table below shows. The use of indeterminate equations seems to be characteristic of Nārāyaṇa.

GSS Rule 1 = GK Rule 2

GSS Rule 3 $\Rightarrow$ GK Rules 1 and 5

GSS Rule 4 = GK Rule 3

GSS Rule 6 $\Rightarrow$ GK Rule 6

GSS Rule 7 = GK Rule 7

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