Introduction

Bernard Bolzano (1781-1848) was born and raised in Prague. Sometimes called the "Bohemian Leibniz", he had the range and depth of knowledge, the energy, and the political engagement to merit the name. He studied philosophy, mathematics, and theology at the Charles University in Prague, and was appointed to a chair in the science of religion there in 1804. In this highly public position, which he had actively sought, Bolzano became widely known as a voice for social justice, a determined and intelligent advocate for freedom of religion, for the rights of the Czech-speaking majority of Bohemia, and a relentless critic of the widespread abuses of power by those in positions of authority. In those days, conservatives and absolutists were prominent in the administrations of both state and church and Bolzano's views, despite their reasonableness and moderation, landed him in a good deal of trouble. Towards the end of 1819, he was removed from his university post on the Kaiser's command, and a lengthy, sometimes absurd, ecclesiastical trial followed.[1] Although Bolzano was eventually cleared of all charges in 1825, he was nevertheless forbidden to teach, effectively prevented from publishing his works in Austria for many years, and subjected to the usual forms of official harassment.

Known in his own time primarily as a social and religious reformer, Bolzano was also a philosopher and mathematician of great talent. As a young man, he decided to undertake two projects of immense scale: the creation of a new methodology, or logic, capable of dealing with modern science, especially mathematics, and of a systematic treatment of all the branches of mathematics according to the requirements of this new logic. During his years at the University he devoted many of his spare hours to logical and mathematical research, and by 1817 had produced significant mathematical contributions, among them *The Binomial Theorem*, which contained one of the first rigorous treatments of the theory

[1]The trial has been documented by E. Winter, *Der Bolzanoprozess* (Brno, 1944). See also W. Künne, "Die theologischen Gutachten in den Verfahren gegen den Professor und den Priester Bolzano," pp. 149-190 in W. Löffler ed., *Bernard Bolzanos Religionsphilosophie und Theologie* (Sankt Augustin: Akademia, 2002); and Bolzano's own account in *Lebenbeschreibung des Dr. Bernard Bolzano* (Sulzbach, 1836).

of power series, and the *Purely Analytic Proof*, a landmark in the development of the foundations of the calculus.[2] From this period, too, dates Bolzano's first published work on logic, part two of his *Contributions to a Better-founded Presentation of Mathematics*.[3]

It is clear that Bolzano was not satisfied with the logic set out in the *Contributions*. Already in 1812 he had resolved to write another treatise on logic,[4] a project he worked on for close to a decade following his dismissal. The result was one of the great works of nineteenth-century philosophy, the *Theory of Science*.[5] The *Theory of Science* was ready for the press by 1830, at which time Bolzano started writing the *Theory of Magnitudes* [*Größenlehre*], a treatise intended to supply a unified foundation for all of contemporary mathematics.[6] Although he wrote several hundred pages, many of them all but ready for the press, but died before finishing his work. Some elements of his mathematical system were published after his death in the *Paradoxes of the Infinite*,[7] edited by his friend and student F. Přihonský, but the bulk remained all but unknown until well into the twentieth century. The essay "On the Mathematical Method", translated here, formed part of the introductory matter of the *Theory of Magnitudes*, and was intended to present the essentials of Bolzano's logic to a mathematical audience.

Due in large part to his troubles with the Austrian authorities, the *Theory of Science* remained unpublished for almost a decade (it was

[2]B. Bolzano, *Der binomische Lehrsatz usw.* (Prague, 1816); B. Bolzano, *Rein analytischer Beweis des Lehrsatzes, daß zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (Prague, 1817), English translation by S. B. Russ in W. Ewald ed., *From Kant to Hilbert: A sourcebook in the foundations of mathematics*, 2 Volumes (Oxford: Clarendon Press, 1996).

[3]*Beyträge zu einer begründeteren Darstellung der Mathematik* (Prague, 1810). English translation by S. B. Russ in Ewald, *Op. Cit.*

[4]*Philosophische Tagebücher 1811-1817*, in J. Berg, F. Kambartel, J. Loužil, B. van Rootselaar, and E. Winter ed., *Bernard Bolzano-Gesamtausgabe* (hereafter **BBGA**) (Stuttgart-Bad Cannstatt 1969–) Series 2B Vol. 16/1, p. 34-36.

[5]*Wissenschaftslehre* (Sulzbach, 1837). New edition by Jan Berg in the **BBGA**. Hereafter **WL**.

[6]**BBGA** IIA, Vols. 7-10; Volumes 7, 8 and 10/1 have already been published.

[7]*Paradoxien des Unendlichen* (Leipzig, 1851), English translation by D. Steele (London: Routledge and Kegan Paul, 1950).

finally published outside Austria, in Bavaria, in 1837). This did not mean, however, that Bolzano's logic remained unknown. Long accustomed to an invasive and often arbitrary censorship, Bohemian intellectuals had developed unofficial channels for communicating their ideas, a precursor of the *Samizdat* system which was later to flourish in that land. Bolzano's mature logic received its first airing in this way, when, in 1833, he had a copy made of the essay on mathematical method and sent it to Franz Exner, the newly appointed professor of philosophy in Prague.[8]

Exner (1802-1853) was born and educated in Vienna, where he studied philosophy with Rembold, who like Bolzano (and for similar political reasons) had been removed from his university chair in 1825. In 1830, Exner was put in the uncomfortable position of taking his teacher's place, being called upon to fill the vacant chair on a temporary basis. In 1832 he moved to Prague, where he was named to the chair of philosophy. Outside of his official duties, in good Austrian fashion, he organized a "circle" of intellectuals which met regularly at his house. Although a born and bred Viennese, he was sensitive to the special circumstances of Bohemia, particularly to the disadvantaged situation of the Czech majority. Politically, though not always philosophically, he was very much on Bolzano's side: with the Bohemian enlightenment and opposed to the conservative reaction in both church and state. Exner was a follower of Herbart, who had a substantial following in Austria at the time, and whose doctrines were to become in effect the official philosophy of the Empire, in part due to Exner's influence when he worked for the Ministry of Education from 1845 until his death.[9]

Exner responded to Bolzano in June of 1833, beginning a correspondence that would continue for the rest of Bolzano's life. The most intense philosophical exchanges occurred during 1833 and 1834, when the letters translated here were written. The two continued their discussion in person in 1834, when Bolzano returned to Prague from June to

[8]Three versions of the essay on mathematical method survive. Our translation is based upon the latest version. Most likely, however, the version that Exner received differed somewhat from this one. A Czech translation of an earlier work on logic, "O logice" (= "Etwas über Logik" **BBGA** 2A5, p. 139-168), was actually published somewhat earlier (1831).

[9]Cf. *Biographisches Lexicon des Kaiserthums Oesterreich*, part 4 (Vienna, 1858); *Allgemeine deutsche Biographie*, vol. 6 (Leipzig, 1877).

November. There would also be a later exchange of views in a pair of papers read at the Royal Bohemian Academy of Sciences in the early 1840s. We have translated Bolzano's contribution, which contains the relevant passages from Exner's, in this volume.

Exner was in many ways a good correspondent for Bolzano. Well disposed towards him, and aware of the importance of some of his contributions,[10] he nevertheless had great difficulty in understanding Bolzano's fundamental concepts—a reaction that would be repeated by later philosophers such as Brentano, Twardowski, Kerry, and to a certain extent also Husserl. Consistently, occasionally maddeningly, unable to grasp certain points, Exner forced his older colleague to explain his views in greater detail for an audience completely unprepared for them.

It is clear from a number of indications that the Bolzano-Exner debate was anything but private. Remarks in the letters indicate that others were keenly observing what the two had to say, reading the letters, and participating in the conversations in Prague. There is also evidence outside the letters that shows that their contents were well-known, at least in Bolzano's circle.[11] After the deaths of Bolzano (1848) and Exner (1853), it seemed that it might become more widely known when Robert Zimmermann, who had been a student of both men, announced his intention to publish their correspondence.[12] He never got around to this task, just as, in general, he did little with the mass of papers Bolzano had entrusted to him. The letters were eventually deposited in the Archive

[10]Exner seems, for instance, to have appreciated the importance of Bolzano's variation logic. Commenting on Drobisch's attempts to construct a logical calculus, he writes: "progress in logic can be achieved not only by an application of the mathematical calculus as made by Drobisch, but also by extending the purely logical calculus. We find such an attempt in Bolzano's *Theory of Science*, ... In indicating the variable components of concepts and propositions, Bolzano inaugurates a manner of consideration similar to the mathematical theory of functions." F. Exner, "Über Leibnitzens Universal-Wissenschaft," *Abh. kön. böhm. Ges. Wiss.* 5 Folge 3 (1845), p. 39.

[11]In Přihonský's *Neuer Anti-Kant, oder Prüfung der Kritik der reinen Vernunft nach den in Bolzano's Wissenschaftslehre niedergelegten Begriffen* (Bautzen, 1850), for example, one can find passages taken verbatim from Bolzano's letters to Exner.

[12]*Akademische Monatschrift.* Würzburg, October 1853. Cited after E. Winter, "Vorwort" to *Bernard Bolzano's Schriften*, Bd. 4: *Der Briefwechsel B. Bolzano's mit F. Exner* (Prague: 1935), p. iv.

of the National Museum in Prague, and it was only in 1935 that Eduard Winter published an edition of the correspondence.[13] A new edition by Edgar Morscher is soon to be published as part of the critical edition of Bolzano's writings.[14] Professor Morscher has most kindly made his new transcription of the correspondence available to us, and we have used it in the preparation of this translation.

Bolzano's logic

Kant, notoriously, wrote that logic was a completed science, one that had not taken a step forwards or backwards since the time of Aristotle. While this claim was false (many steps, both backwards and forwards, had been taken in the interim), it was fairly widely accepted among Bolzano's contemporaries, with some justification. Still following Aristotle's pattern, the majority of logical treatises of the time began with a study of terms (representations, ideas), continued with a discussion of judgments (or propositions), of inferences (syllogistic), and of the organisation of the sciences. Greater emphasis, perhaps, was placed upon the theory of ideas by modern logicians, with a corresponding diminution of the space allotted to the later parts of logical theory. Overall, though, logical theory might well have seemed not to have changed in the course of two millennia: on all sides it was still claimed that terms were of two sorts, singular or general; that the basic form of judgments was subject-predicate; and that all inference was syllogistic.

It was against this stagnant background that Bolzano presented his new logical theories. Like Frege, he saw that fundamental changes in the traditional approach to logic were necessary, especially if, as he planned, logic was to be the guiding light for the construction of an adequate foundation for mathematics. Like Frege, too, Bolzano broke with logical tradition on the following three points: 1) He recognized senses (Bolzano calls them "objective propositions and ideas" or "propositions and ideas in themselves") as abstract objects, constituting an independent "third realm" (Bolzano: "the collection of propositions and ideas in themselves"), a move which had the effect of detaching formal logic from psychology; 2) he reversed the customary order by defining ideas

[13] *Ibid.* Despite its great merits, Winter's edition is unreliable in places.

[14] **BBGA**, Series III, Volume 4 (forthcoming).

in terms of propositions; finally, 3) he introduced function-theoretical considerations into logic, using a technique of variation to define relations such as consequence, equivalence, and probability. In these and other respects, Bolzano was well ahead of his contemporaries, and to the extent that we accept the standard historiographical categories, we should probably call his logic post-Fregean.

Bolzano's view of logic went hand in hand with his view of mathematics. Based on a careful analysis of the content of classical mathematics, he concluded that the most basic mathematical object was what he called a *collection* or *system* [*Inbegriff*], that is, a structured set of elements, and that all of mathematics could be elaborated as a development of the theory of collections. In the parts of the *Theory of Magnitudes* following the essay on method, Bolzano presents his theory of collections, distinguishing such varieties as sets, sums, multitudes, and sequences.[15] The goal of his foundational research was to construct axiomatic theories of the various mathematical structures on this basis. Geometry, for example, he developed as a theory of collections of points possessing metrical and topological relations, analysis as a theory of quantities possessing relations of order, functional connections, etc. But Bolzano saw that his basic conception also applied to logic: not only were the objects of mathematics structures, mathematics *itself* (and in general any science) was a structure, namely, a collection of *true propositions* (themselves structured entities) ordered by the relation of ground-consequence. The same sort of foundational analysis used in mathematics, therefore, was called for in logic. The formal core of logic, that is, is mathematical, the theory of the particular kind of collections called sciences.

Just as mathematical objects seem to have a certain independence and objectivity, so too, Bolzano thinks, do sciences. Our minds may come to understand, for example, that a certain result follows from certain assumptions, but it is not our understanding that makes it so. Rather, there is an intrinsic, objective relation between these assump-

[15] *Größenlehre*, **BBGA** IIA 7, p. 98 f. Bolzano's theory of collections is also summarized in the *Paradoxes of the Infinite*. For discussion of the theory, see J. Sebestik, *Logique et mathématique chez Bernard Bolzano* (Paris: Vrin, 1992), p. 305 ff.; P. Simons, "Bolzano on Collections," *Grazer philosophische Studien* **53**(1997)87-108.

tions and the result that the mind discovers. Bolzano thus breaks with the widespread view that mathematics is or should be organized in accordance with human cognitive powers. For many of Bolzano's contemporaries, an axiom was an epistemologically primitive proposition, one that was so luminously evident that there existed nothing more obvious from which it could be proved. For Bolzano, by contrast, an axiom is a logically primitive proposition, one that is a ground but never a consequence of other propositions. A proposition can be perfectly obvious and still require proof. Bolzano took particular pleasure in finding proofs for obvious propositions, since they often reveal unnoticed but vital elements of the intrinsic structure of the sciences to which they belong. On the other hand, axioms in his sense (or *basic truths*) may be far from obvious; as he remarks, they might well seem false until we have enough insight into how they support other, known, truths.[16]

Bolzano's conception of logic is considerably wider than most today. For him, logic is just methodology, the theory of science, and its goal is to tell us how to organize all of human knowledge and present it in treatises. Because of this wider scope, Bolzano discusses many topics in his logical works that would not today be considered to belong to logic. The third volume of the *Theory of Science*, for example, is a treatise on epistemology and heuristics, and the fourth contains, among other things, a manual of scientific composition.

Logical objects (*Method*, §1)

The independence and objectivity of sciences finds expression in Bolzano's notion of *propositions and ideas in themselves*. He insisted on the necessity of distinguishing individual thoughts, be they judgments or ideas, from what he called "propositions and ideas in themselves" or "objective propositions and ideas." His reasons for so doing are straightforward. First of all, logicians commonly acknowledge that different people can entertain the same proposition, e.g. Goldbach's conjecture. So too, a single person may think the same proposition at different times. Finally, the same proposition might occur in quite different forms: judged to be true or false, merely entertained, expressed verbally in various languages, etc. Bolzano points out that what is identical

[16] *Beyträge zu einer begründeteren Darstellung der Mathematik* (Prague, 1810), II, §21.

in such cases cannot be the actual thoughts or expressions, which are certainly numerically distinct. It must be, instead, the content, or meaning, of their thoughts that is the same. It is this content, or meaning, that he calls a proposition in itself or an objective proposition. A subjective proposition, by contrast, can be said to be the result of some individual *grasping* the objective proposition, though "grasping" has to be understood here in a purely metaphorical manner. While subjective propositions are parts of reality, existing at certain times in individual minds and having causal relations, their objective counterparts have no real existence, lack spatial and temporal determinations, etc. Bolzano prefers to say that *there are* propositions (thus he quantifies over these abstract objects), but that propositions do not have *actuality*. Thus he distinguishes two uses of the term "existence"–one identical to the modern use of the existential quantifier (existence in this sense is a property of ideas, namely the property of having objects, which Bolzano calls *objectuality* [*Gegenständlichkeit*]), the other (actuality) to designate an attribute of objects in the real world.

Bolzano's commitment to the existence of propositions in themselves does not seem to have been overly strong. In the *Theory of Science*, for instance, he claims that one would be justified in admitting the existence of propositions in themselves as soon as one recognizes that it is useful to speak of them —thus he appears to espouse a pragmatic approach to ontology (at least with respect to some abstract objects).[17] We speak of propositions, we quantify over them, and it is difficult to see how logic could be pursued without doing so (or at least without recourse to some similar expedient). This, for Bolzano, seems to be justification enough for admitting them into our ontology.

Propositions and ideas (*Method*, §§2-5)

As we remarked above, Bolzano departed from logical tradition in defining ideas in terms of propositions rather than the other way around. This decision is a surprising one. For Bolzano clearly thought of propositions as structured entities, and one would have expected that, in line with his methodology, he would attempt to say what propositions were in terms of their parts and the relations between these parts. This approach had been very fruitful in mathematics. Why not use it here?

[17] **WL**, §20.1.

Bolzano tells us that he was unable to find a definition for the concept of a proposition,[18] offering instead what he calls an "explication" of the term. That is, instead of attempting to produce an expression synonymous with "proposition", he shows us how to use the term correctly. We will better understand his decision when we reflect on the task set by the question posed above. Consider a natural language such as English. Suppose that somehow we were able to put together a list of all the words of the language, or at least an exhaustive description of possible kinds of words. The task of defining propositions would then be very much like that of providing a universal grammar, that is, specifying a set of rules which would determine exactly which combinations of words would constitute well-formed sentences, but with this difference: the rules Bolzano seeks would be those underlying not just one language, but any possible language. Small wonder he could not see his way clear to an adequate definition.

What, then, is a proposition? Writing to Exner, Bolzano says that by carefully considering the following sentences, it should be possible to understand what he means by the term:

1. A proposition is exactly one of the two, true or false.

2. A proposition does not have actual existence.[19]

A proposition, that is, is an abstract object, a truth-bearer without actuality.

Given the concept of a proposition, the concept of an *idea* can be defined as a part of a proposition that is not itself a proposition.[20] Thus a proposition like "Any native-born American can become President of

[18]**WL**, §23.

[19]Bolzano to Exner, 22 November, 1834; below, p. 141.

[20]In the **WL**, §52, Bolzano writes that this characterization, though it suffices to fix the extension of the term "idea", should not be considered a definition. In a letter to Exner of 22 November, 1834 (below, p. 145), though, he seems to be on the verge of changing his mind: "As concerns the concept of an *idea* in particular, I am inclined to believe that the explication 'an idea is a part of a proposition that is not itself a complete proposition' is the true *definition* of this concept. For it is quite common for the concept of a part to include the concept of the whole. Thus for example, in the concepts of dial, hand, etc., the concept of the entire clock surely occurs. For we certainly think by dial simply 'a surface inscribed with numbers in such a way that it can be used in a *clock* in

the United States" might be said to contain a variety of sub-propositional parts or ideas, e.g. "any", "can" "become" "President of the United States", but also "the", "of the", etc. Note that ideas may themselves have parts.

Bolzano calls the set of parts of an idea (or of a proposition) its *content*, and remarks that the same content may be combined in different ways to produce different ideas, e.g. "number greater than two and less than four", "number greater than four and less than two". An idea with no parts is called *simple*. Furthermore, some ideas have, or represent, objects (e.g. "mammal", "prime number"), while others do not (e.g. "round square", "and"). When an idea has objects, the set of all such objects is called its *extension*. An idea with a non-empty extension is said to be *objectual* (or, more precisely, to have the property of *objectuality*), the rest are called *objectless*.

Intuitions and concepts (*Method*, §§6-7)

In §6, Bolzano challenges another commonplace of traditional logic, namely, that all simple ideas must be *general* ideas. Kant, for instance, wrote in his *Logic* that concepts are one and all (simple or complex) *characteristics*, and that it was a *tautology* to speak of general concepts. By this he did not intend to exclude the possibility that a characteristic might happen to be found in just one object. Rather the point was that even if this were so, it would still be possible for it to occur in a second object (i.e. the mind might have occasion to construct another object having that characteristic).

Some arguments were advanced in support of this generality thesis. Leibniz had claimed in a well-known essay of 1684,[21] for example, that the simple concepts were none other than the absolute attributes of God, also known as perfections. Perfections, he maintained, are by their very nature compatible in any combination (something he appealed to in his improved version of the ontological argument). That is, if

order to ... etc.' So too the concepts 'head', 'neck', 'heart', 'lung', etc. cannot be defined without mentioning the relations in which these parts stand to the whole organism; i.e. in the concept of the part there occurs the concept of the whole that the part belongs to."

[21] "Meditations on knowledge, truth, and ideas," tr. L. Loemker in G. W. Leibniz, *Philosophical Papers and Letters* (2nd ed. Dordrecht: Kluwer: 1989).

$p_1, p_2, p_3, \ldots$ are perfections, there is a possible object for any combination of these attributes. In particular, suppose that a possible object has the attributes $p_i, p_j \ldots, p_n$. Then there will also be a possible object that has the attributes $p_i, p_j \ldots, p_n$ and p_{n+1}, but also a possible object that has $p_i, p_j \ldots, p_n$ but lacks p_{n+1}. Having incompatible properties, these possible objects are clearly distinct. That is, for any given simple idea, there are many possible objects possessing it; i.e. it is general in character.

Others, like Exner, presented arguments based upon a dispositional account of representation. An object acts upon me, Exner claimed, stimulating a particular mental state (a simple idea), which may be applied to that object. But the very same mental state might well have been stimulated by another object. Thus, he argued, all simple ideas are general in character.

It will be noticed that both of these arguments, even were one inclined to accept their far reaching assumptions, just ignore the problem of simple ideas that have only one object and cannot possibly have more than one. In vain, however, did Bolzano point this out to Exner, who seems to have been incapable of understanding how this might be possible. Seeing no compelling reason to think simple ideas with exactly one object impossible, Bolzano once again decided to go his own way. He called such ideas *intuitions*. Ideas which are not intuitions, and which contain no intuitions as parts, are called *concepts*; while ideas containing both intuitions and concepts are called *mixed*.

It is all well and good to establish that there are, or at least can be, ideas *in themselves* that, although simple, represent exactly one object. But it might well turn out that this abstract possibility was of little interest if human beings never in fact had or could have such ideas. Bolzano recognized this, and tried to show that human thought necessarily involved intuitions. What is more, he sought to show that the existence of such ideas was tacitly assumed by many if not most modern philosophers. This was especially evident in the case of judgments of perception, so for the moment we will focus on these.

Consider Descartes' influential account of perceptual judgment. The mind is constantly affected in certain ways by objects external to it; it has sensations of heat, colours, and so on. By themselves, these sensations present objects at best confusedly (many authors said, not at all). I

say I see a tree, but more carefully considered, I can only say that certain sensations occur in me. It is the work of the mind, Descartes observes, to *judge* that there is an extramental object causing my sensations, and to make further judgments about the constitution of that object based on the patterns I find in them.

Bolzano's own account of perception is similar in broad outline, but differs in respect of semantic precision. He does not question the view that physical objects are experienced mediately, by means of sensations, but invites us to consider more carefully the structure of the judgments made in such cases. Let us suppose, for instance, that we are standing before a rose. What then happens is that certain sensations arise within us: an experience of a certain shade of red, of a pleasant fragrance, and so on. The mind then makes judgments concerning these sensations, for example, that they were all caused by one and the same physical object.

Already at this first step, however, we encounter a question which requires an answer before we can proceed, namely: are sensations *ideas*—a view shared by many philosophers, among them Descartes, Locke, Kant, and many others, and reflected in the choice of the widely used term "sensory ideas"? For Bolzano, there was but one way to answer this question, namely, to determine whether sensations can be parts of subjective propositions. Now in his view, a sensation can be the *topic* (or *subject*) of a judgment—that is, can be the object which the judgment concerns or is about—but it is a mistake to consider a sensation one of the *terms* of a judgment. We judge that a certain sensation which we had was caused by a certain physical object; but the sensation itself is no more part of that judgment than the physical object is. Rather, both are referred to, or represented by, terms of the judgment. "The sensation itself is one thing," as he had observed in the *Theory of Science*, "the idea of the sensation quite another. . . ."[22]

Bolzano continues by inquiring into the nature of the ideas of sensations that enter into such basic judgments of perception. We tend to speak of sensations in general terms, i.e. of seeing a certain *shade* of colour, or of feeling a certain *kind* of pain (or simply feeling *pain*) which we experience as the result of a physical object acting upon us. Many philosophers seem accordingly to have assumed that the ideas caused in us by the action of external objects were general ideas—thus,

[22]**WL**, §35.8.

for instance, Aristotle's well-known formulation, later elaborated by Brentano: "...though the act of sense perception is of the particular, its content is general...."[23] Perhaps some thought that the concreteness of the token (which came to exist in the mind because of the action of the object) explained how the idea, though general in character, *on this occasion* represented only to a single object. But from Bolzano's point of view, this was a non-starter: ideas, for him, are individuated by their extensions. Where the extensions differ, so too do the ideas.[24] There is simply no such thing as an idea that is general in itself but singular in use.

For Bolzano, it was clear that it is not a general term that is required in such basic judgments. For it is not pain *in general* which is caused by the action of a physical object, but rather a particular experience of a certain kind of pain, the sensation of pain which I experience on that occasion. As he wrote to Exner:

> Someone brings a rose near us. We see—not red in general, but *this* red present in the rose. We smell, not odour in general, but rather only *this* pleasant fragrance, which just this rose has; when we are injured by one of its thorns, we feel not pain in general, but rather this determinate pain.[25]

Thus what we need is an idea which has the particular experience as its only object. This idea cannot be a general term like "pain" or "red"; rather, it is an idea that is *subordinate* to these general terms. The structure of these quite basic judgments of experience, Bolzano thinks, is thus more adequately expressed as follows:

> (1) This (which I am presently experiencing), which is a pain, was caused by a certain physical object.

The general term "pain" may occur in such judgments, he observes, but cannot be the subject idea. Indeed, a judgment could perfectly well bypass the qualification "which is a pain", and assign the cause of the experience directly to an object.

[23] *Posterior analytics* II, 19.

[24] Cf. Bolzano's letter to Exner of 22 November, 1834, below, p. 147.

[25] Bolzano to Exner, 9 July 1833; below, p. 96.

> (2) This (which I am presently experiencing), was caused by a certain physical object.

While the qualification, for its part, might be brought into play through another judgment subsuming the particular experience under the general concept of pain:

> (3) This (which I am presently experiencing) is a pain.

The subject idea in these cases, which is necessarily singular, cannot be merely a general or descriptive term, something reflected in the inexpressive demonstrative: "this".

Bolzano was thus satisfied that such basic judgments concerning sensations required ideas with exactly one object. In order to establish the possibility of subjective intuitions, then, it remained to show that at least some of the ideas occurring in judgments of this kind could be *simple* as well. There are, in effect, two separate lines of argument here.

The first one, somewhat disappointing, argues for the simplicity of the subjective intuition based on its causal pedigree: the rose affects the mind, producing a sensation, and the sensation produces, as its immediate effect, the intuition which represents it. Because the intuition is the "next and immediate effect" of the sensation, he argues, it must be simple–a surprising argument, as Kerry remarked, from a mathematician celebrated for his analysis of continuity.[26]

The second, semantic, line of argument is more promising. Here Bolzano asks us to consider the subject concept "this, which I just now see", and argues that the idea expressed as "this", alone and unaided, must do all the work of singling out the object. The remaining parts are thus, in his terminology, *redundant*. He reasons as follows:

> [A]s certain as it is that ideas of the form "This, which now occurs in me" are singular ideas, just so is it certain that among these there are at least some which are completely *simple*. For if we suppress the thought of any additions like "which occurs in me just now", "which I just now see, hear,

[26]B. Kerry, " Ueber Anschauung und ihre psychische Verarbeitung," *Vierteljahrsschrift für wissenschaftliche Philosophie*, **9** (1885), p. 434.

> or feel", [...][27] etc., the bare idea designated by the word "this" is certainly a completely simple idea. But the object that it represents remains throughout the same single one, whether we think the additions or not. For, if we consider them more closely, all these additions express no more than certain attributes which that single object which we just now represent possesses precisely because it is this one and no other; indeed, our idea does not become restricted to that single object only by means of these additions, but rather becomes redundant through them.[28]

That is, the circumstances that it is *I* who am experiencing this pain, and that I am experiencing it *now*, follow from the circumstance that it is *this* and not something else. For the idea "I", in Bolzano's view, is just that of "that something that is conscious of certain ideas"[29]—thus in order to represent the particular I that experiences *this*, I will already need to have the idea *this*. "Now", for its part, is not specific enough in any case (even supposing it could be determined without recourse to "this"), for Bolzano would not accept that the mind only experiences one thing at a time. The same holds for ideas such as "this red", "this sound" and the like, for the additions "red" and "sound" are general terms, and cannot by themselves restrict the extension to a single object:

> [W]hen any real object through its activity produces an idea of the kind: "*this* (red)", "*this* (pleasant smell)" and the like, the part of this idea that we designate with the word "*this*" is, or at least can be, simple, since all further determinations like $a, b, c, \ldots$ are only *redundancies*. For all of these determinations already follow from the circumstance that it is just this and no other thing that we now represent. Accordingly, our idea does not cease to refer exclusively to this object (does not cease to be a singular idea) if we omit

[27] Bolzano adds here "which I am now pointing at with my finger"—this seems to be a mistake, given that he elsewhere claims that all objects of intuitions are mental occurrences—one can scarcely point at one's pain, for instance.

[28] *Method*, §6.4; below, p. 51.

[29] **WL**, §44.2.

these determinations.[30]

The bare "this" seems to be doing all the work. It seems to have no descriptive content. Any additions to it seem destined to be redundant. Why not, then, suppose that "this" as it occurs in such contexts is indeed a simple idea? Bolzano thus seems to have found something quite like Perry's "essential indexicals" at the heart of judgments of perception—for it seems clear from Bolzano's analysis that any expression of such judgments will contain ineliminable demonstratives or indexicals.[31]

In the essay on mathematical method, the most important use made of the theory of intuitions is the division of propositions and sciences into the purely conceptual and the empirical.[32] An idea is called a *concept* iff it is not an intuition and contains no intuitions as parts. Propositions that contain no intuitions as parts are called *purely conceptual*; all the rest, those containing at least one intuition, are called *intuitional* or *empirical*. Since a science is an ordered collection of propositions, one can also speak of purely conceptual and empirical sciences. The central parts of mathematics—among others, the theory of collections, number theory, algebra, analysis, and geometry—are then claimed to be purely conceptual sciences. Neither their propositions, nor their proofs, contain even a single intuition.

The distinction between purely conceptual and intuitional propositions and sciences effectively replaces (even if it does not exactly coincide with) the standard distinction between *a priori* and *a posteriori*. The important distinction is that between propositions that contain direct reference to particulars, and those that do not; or, from another point of view, between cases where demonstrative (and indeed—in the case of subjective ideas—causal) reference to particulars is involved from cases where it is not.

[30]Bolzano to Exner, 9 July 1833; below, p. 98.

[31]J. Perry, "The problem of the essential indexical," in *The Problem of the Essential Indexical and other Essays* (Stanford: CSLI Publications, 2000). See M. Textor, *Bolzanos Propositionalismus* (Berlin: DeGruyter, 1996) for a thorough examination of the connections between Bolzano's theory and the modern semantics of indexicals and demonstratives. The concept of an intuition is even more closely related to Russell's notion of a "logically proper name".

[32]In the **WL** (§182), Bolzano also uses the concept of an intuition to define modal concepts such as necessity and possibility. Cf. below, p. 111.

Variation logic (*Method*, §8; cf. WL, §§147-48, 155 ff)

§8 Presents a brief sketch of Bolzano's logic of variation, perhaps the best known of his logical theories due to its rather close affinities to the accounts of logical consequence given by Tarski and of logical truth given e.g. by Quine.

The basic insight underlying Bolzano's theory is both straightforward and familiar. Consider, for example, a syllogism like:

> No whales fly. Some mammals fly. ∴ Some mammals are not whales.

We might say that this argument has a *form*, namely (replacing "whale", "thing that flies", "mammal" with the place-holders A, B, C):

> No A are B. Some C are B. ∴ Some C are not A.

And we might say that this form is valid, meaning by this that whenever we substitute terms of the appropriate sorts for A, B, and C (here, class terms), if it should happen that the premises all turn out to be true, it will also turn out that the conclusion is true. Now while many logicians of Bolzano's or earlier times might have been capable of making such an observation, very few had noticed that such traditional syllogistic forms are a vanishingly small part of the collection of valid inferences,[33] and that this description of validity can be extended to a great many other forms of argument. Consider, for example, the following:

$$\pi > 3, 3 > e \therefore \pi > e$$

This argument might also be said to have a valid form in the above sense, namely (replacing all occurrences of $\pi, 3, e$ by the placeholders x, y, z):

$$x > y, y > z \therefore x > z$$

Here too, whenever we replace x, y, z with ideas of the appropriate sort (ideas of real numbers), we find that whenever the premises all come out true, so too does the conclusion.

Actually the above argument might be said to have several valid forms on the above understanding, among them:

[33]Leibniz, of course, is a notable exception. See for instance his many excellent remarks in book 4 of the *New Essays*.

$x > 3, 3 > e \therefore x > e$

where only "π" is considered variable, and

$\pi > y, y > z \therefore \pi > z$

where both "3" and "e" are.

So, too, the following argument has a valid form:

J. Fred Muggs is a primate. $\therefore$ J. Fred Muggs is a mammal.

Namely

x is a primate $\therefore$ x is a mammal

Bolzano takes account of this multiplicity of forms by defining *deducibility* as a triadic relation, viz. a relation involving premises $(A, B, C, \ldots)$, conclusions $(M, N, O, \ldots)$, *and* the variable elements $(i, j, k, \ldots)$. His definition is thus more general[34] and arguably more useful than those commonly encountered in introductions to logic, where all non-logical terms are always considered variable, and where, in order to declare an argument form like $x > y, y > z \therefore x > z$ valid, one has to add meaning postulates or additional axioms, or supply a "missing" premise. Bolzano's account of validity is especially congenial to mathematicians, who usually view things in just the way he describes, inferring $\pi > e$, for instance, directly from $\pi > 3$ and $3 > e$ without thinking that anything is missing.

Bolzano begins his formal treatment by giving a general definition of the relation of *compatibility* (akin to the concept of consistency), which may hold between various propositions (note that his theory is developed with propositions and ideas rather than with linguistic expressions). The propositions $A, B, C, \ldots$ are said to be *compatible* with the propositions $M, N, O, \ldots$ with respect to the variable ideas $i, j, k \ldots$ iff there are ideas $i', j', k', \ldots$ which, when substituted for $i, j, k, \ldots$ make all of $A, B, C, \ldots, M, N, O, \ldots$true. For example, the propositions:

[34]In contrast to the usual historical sequence, Bolzano's definition is actually a generalization of Tarski's.

Joe is older than John.

John is taller than Joe.

are compatible relative to the designation of all occurrences of "Joe" and "John" as variable, for substitutions can easily be found that will result in both propositions being true. On the other hand, the propositions

Joe is older than John.

John is older than Joe.

are not compatible with respect to the same variables (though they are compatible if only the part "is older than" is considered variable).

With the definition of compatibility in hand, Bolzano proceeds to his well-known definition of *deducibility*:

> If one or more propositions $A, B, C, \ldots$ are compatible with one or more others $M, N, \ldots$ with respect to the components $i, j, \ldots$, then there must be, as just said, at least some ideas that, when put in the places of $i, j, \ldots$ make all of $A, B, C, \ldots$ as well as all of $M, N, \ldots$ true. One especially noteworthy case occurs, however, if not just some, but *all* of the ideas that, when substituted for $i, j, \ldots$ in $A, B, C, \ldots$ make all these true, also make all of $M, N, \ldots$ true. In this case I say that the propositions $M, N, \ldots$ stand in the relation of *deducibility* to the propositions $A, B, C, \ldots$ with respect to the variable parts $i, j, \ldots$.[35]

Equivalence is then defined as mutual deducibility relative to a set of variands.

We see that Bolzano's definition of consequence differs from modern ones in that he requires that the premises be compatible relative to the specified variables—thus nothing follows from incompatible premises in his system.

Bolzano does not think that deducibility as he has defined it is the last word on the relations between propositions in an axiomatic system.

[35] *Method*, §8.2; below, p. 54.

To begin with, he distinguishes between the quite general sense of deducibility given above and a narrower sense, where, in addition to the conditions specified above, the conclusion is not deducible from any proper subset of the premises $A, B, C, \ldots$. This narrower notion of deducibility, investigated at some length in the *Theory of Science* under the name of "exact (or irredundant) deducibility", combines the general notion with the requirement that none of the premises should be idle or redundant.[36] Further remarks are added in §12, devoted to the concept of proof. There, among other things, Bolzano makes it clear that the intrinsic ordering of propositions in an axiomatic science is not determined solely by relations of deducibility, not even by the narrower relation of exact deducibility, but depends as well upon such factors as the content and generality of propositions (see also §17).

Explications and definitions (*Method*, §9)

Bolzano devotes §9 to a discussion of "Explications", ways of communicating the meanings of expressions. The use of explicit definitions, well understood by mathematicians, is discussed briefly. But this means, Bolzano notes, is not always available. We cannot, for example, define simple concepts, nor can we give definitions when we lack the words to express the part-ideas of which some concept is composed. In such cases, other means will have to be used. Bolzano points out one of particular interest, a form of contextual definition. He describes this as follows:

> We set out various propositions in which the concept that is designated by our sign appears in such combinations that no other concept could be thought in its place if these propositions were to express something reasonable. By considering and comparing these sentences the reader will gather by himself the meaning of our sign.[37]

A detailed example of this method may be found in Bolzano's response to Exner's essay on nominalism and realism, where it is used to convey the meaning of the expression "object of an idea" (pp. 179-181 below).

[36] **WL**, §155.26 ff.
[37] *Method*, §9; below, p. 57.

Foundations of mathematics (*Method*, §§10-14)

Having set out the basics of his logic, Bolzano proceeds in §§10 ff to develop his views on the foundations of mathematics.[38] §§10 and 11 are devoted to the questions of conceptual analysis and definition. His position is clear: to produce a truly scientific exposition of some branch of mathematics, we should seek explicit definitions for all the concepts that occur in it. This is good practice for two reasons: (1) if we do successfully define concepts, we often find that these definitions permit insights into the science that would otherwise have escaped us; (2) the only way that we can recognise that a concept is simple, and hence indefinable, is by finding that all attempts at defining it in terms of other concepts fail. The result of such attempts, if perfectly successful, will be a system of concepts, beginning with the indefinables (whose meaning will be communicated via contextual definition, etc.), and continuing with explicit definitions of all other terms.

§§12-14 continue with a discussion of the deductive order of axiomatic theories. Bolzano begins by distinguishing two different notions of proof: subjective proofs, which serve the function of convincing someone that a proposition is true, and objective proofs, which display the objective dependence of one truth upon others. This network of relations of dependence is by no means obvious at first glance. Much of it is hidden, never even dreamt of. Only careful, detailed research can bring it to light. The task of the philosophical mathematician (i.e. the foundational researcher) is to do just that, by inquiring into the meaning of the terms and propositions of mathematics, and their *objective* relations of dependence—*their* reasons for *being true* rather than *our* reasons for

[38] A study of Bolzano's views on the foundations of mathematics would be incomplete without consideration of his work in the field. The reader interested in Bolzano's mathematics would do well to begin with the 1817 essay *Purely Analytic Proof* (see above, p. 6, note 2). Further English translations by S. B. Russ of a great many of Bolzano's mathematical writings will soon be published by Oxford University Press (personal communication). Bolzano's mathematics is discussed at length in A. Behboud, *Bolzanos Beiträge zur Mathematik und ihrer Philosophie* (Bern: Bern Studies in the History and Philosophy of Science, 2000); P. Rusnock, *Bolzano's Philosophy and the Emergence of Modern mathematics* (Amsterdam: Rodopi, 2000) and J. Sebestik, *Logique et mathématique chez Bernard Bolzano* (Paris: Vrin, 1992).

believing them to be true. The goal is to arrive at perfectly distinct definitions of the basic concepts of mathematics, exact statements of its propositions, treatises that reflect the objective order of deductive dependence between them, presenting them "in the order they themselves prescribe."[39]

Although many of his contemporaries, following the lead of Kant, were decidedly hostile to Bolzano's project, he nevertheless found support for his views among mathematicians and better-informed philosophers. The ancients had often proven entirely obvious results, and Leibniz had said that it was a good thing to try to prove obviously true propositions. But this awareness was not always entirely sharp, and all too often the subjective and the objective relations were confused. It was said, for instance, that it would be foolish to seek to define terms already clearly known in themselves, and equally foolish to seek to prove the self-evident. These remarks would be just, Bolzano concedes, if what was meant were *subjective* conveyings of meanings or *subjective* proofs [certifications, *Gewissmachungen*]. If something is already obvious, for example, there is admittedly no need to convince ourselves of its truth. But nothing could be more damaging, in his view, than the surreptitious extension of these claims to *objective* definitions and proofs. In his first publication, he had set out a reminder to this effect:

> First of all, I set down as a rule that the obviousness of a proposition never releases me from the obligation to seek a proof for it—that is, until I distinctly understand that and why one cannot demand that such a proof be furnished.[40]

And, as is well known, it was proving the obvious, and defining concepts that "everyone knows and doesn't know", that earned Bolzano the fame he enjoys as a mathematician. Such undertakings have a variety of benefits: not only do we become more clear on just what the theorems say and thus better able to provide valid proofs for them, we also become better able to order and impart our knowledge. Most important, perhaps, is the fact that they bring to light new problems and suggest new directions for research.[41]

[39] **WL**, §2.

[40] B. Bolzano, *Betrachtungen über einige Gegenstände der Elementargeometrie* (Prag, 1804), Vorwort.

[41] *Ibid.*

After presenting his own views on the subject, Bolzano provides detailed criticisms of the views of Kant and his followers on mathematics (see the notes to §11 and §14). One of the earliest refutations of Kant's account of mathematics, this is also one of the best.

The correspondence with Exner

With its many innovations, Bolzano's logic was difficult enough for his contemporaries to understand even with the aid of the detailed explanations given in the *Theory of Science*. Expressed in highly compressed form in the essay on mathematical method, it presented a still greater challenge. Not surprisingly, we find that the correspondence between Bolzano and Exner is devoted in large part to clearing up Exner's misunderstandings, getting clear on just what Bolzano meant. Exner seems to have no quarrel whatsoever with Bolzano's logic of variation and his remarks on mathematical method: "Everything after §6," he writes, "seems to me completely correct."[42]). It is rather the foundations of Bolzano's logic that Exner finds troubling. He has particular difficulty accepting Bolzano's appeal to abstract objects in logic (propositions and ideas in themselves), and with Bolzano's theory of representation, especially what he has to say about intuitions. We will say a few words about the course of the dispute in each case.

Logical objects: propositions and ideas in themselves

We have seen that Bolzano maintained that *there are* propositions in themselves, that is, that the idea "proposition" is objectual, but also that propositions in themselves do not enjoy actual existence. Exner cannot admit such dubious entities:

> I cannot approve when you, although denying existence to objective truths and ideas, nevertheless give them a—I might say—ghostly existence. The objective ideas are supposed to be grasped through the subjective? How can the non-existent be grasped by the existent? What is grasping supposed to mean here?[43]

[42]Exner to Bolzano, 27 June 1833; below, p. 89

[43]Exner to Bolzano, 10 December 1834; below, p. 152.

For him, propositions in themselves (or objective propositions) are nothing other than abstractions obtained from actual judgments, disregarding the psychological context of their occurrence.

Bolzano objects that we often speak of unknown truths. Perhaps no one knows, for example, how many apples were on this tree last year, yet there seems nothing wrong with saying that one of the propositions "There were n apples on this tree last year" for $n = 1, 2, 3, \ldots$ is true, without asking whether anyone has actually thought all these propositions (still less judged them to be true).

Exner replies:

> Every truth exists only in the consciousness of an individual, in an individual understanding, nowhere else and in no other way. If one says that there are countless truths that man has not yet discovered, I reply: there are no such truths, there are only things and their relations; truth only comes into being with ideas, and to speak of "discovery" here is to use a figurative expression. The proposition presupposes someone who proposes, the representation someone who represents.[44]

Confronted with this fundamental opposition, Bolzano tries his best to bring Exner around to his point of view. To begin with he notes the difficulties involved in saying that objective propositions are abstractions from actual *judgments*. On Exner's account, for example, in order to say that the sentence "A triangle is not a triangle" expresses a proposition, we would have to maintain that someone had actually judged that this was so. Not only is such a claim highly dubious, Bolzano remarks, it also seems clear that such questions do not belong to logic at all. Beyond this, he shows that Exner, at least to judge from his words, also accepts that there are objective propositions. Exner interprets the claim that there are unknown truths as follows: "Representing beings haven't yet discovered all the relations among all (or certain) ideas, or between ideas and real things."[45] Bolzano points out to him that the relations of which he speaks are not to be met with in the minds of representing or judging beings:

[44] Exner to Bolzano, 27 June 1833; below, p. 85.

[45] Exner to Bolzano, 11 November 1834; below, p. 136.

> [Here] you change from your concept of a proposition in itself to mine without being aware of it. For now you understand by truth in itself a certain *relation* that holds between the ideas, a relation that is not something existing, and is consequently a mere *proposition in itself* in my sense of the term.[46]

The nature of representation

We have seen that Bolzano reversed the customary order in logic by beginning with propositions and then defining ideas in terms of these. This decision was a typical product of his careful, precise approach to philosophical problems. An apparently minor point, it led in a fairly natural way to a radical break with received opinion.

Traditionally, ideas were discussed first, and because of this the relation between idea and object was often thought to require special explanation, an answer to the question: what makes my idea an idea of just these objects and no others? Often, an answer was sought in the assumption of some sort of similarity between idea and object. Aristotle, for instance, had claimed that the thinking part of the soul was "capable of receiving the form of an object, that is, ... potentially identical in character with its object" and that in cases of actual knowledge, the mind was "identical in form with its object."[47] Leibniz, for his part, had maintained that the mind was a living mirror of the universe, its internal states (namely, ideas/representations) representing the things distinct from it by virtue of a structural homomorphism. When I have an idea, he wrote, "there must be something in me which not merely leads me to the thing but also expresses it." He explains: "that is said to express a thing in which there are relations which correspond to the relations of the thing expressed. But there are various kinds of expression. [...] what is common to them all is that we can pass from a consideration of the relations in the expression to a knowledge of the corresponding properties of the thing expressed."[48] One of Kant's reflections shows

[46] Bolzano to Exner, 22 November, 1834; below p. 144.

[47] *On the Soul*, III.4 ($429^a14f.$), III.5 (430^a14f).

[48] "What is an idea?" tr. L. Loemker in G. W. Leibniz *Philosophical Papers and Letters* (Dordrecht: Kluwer, 1989), p. 207.

that this understanding was part of his intellectual heritage: "*Representation* is that determination of the soul that applies to other things. I say that the representation applies when its features conform to those of the external things," that is, when the representation "is composed out of its component concepts in the same way in which the entire represented thing is composed out of its parts."[49] The many philosophers who thought of ideas as mental pictures of their objects were also partial to this view, for obvious reasons.

Coupled with such views, one often finds the belief that there is no intrinsic relation between an idea and its object(s). Although the mental images called ideas were thought to be stimulated in some instances by external objects resembling them, this was by no means always the case—in memory, in imagination, in the case of illusions, and so on, they might come and go in the most varied rhythms, in no noticeable correlation with what goes on outside the mind. Thus the presence of an idea was in itself thought to be no guarantee of the presence or even the existence of its object. It seemed to some to follow that the relation of idea to object was something that had nothing to do with the idea itself, and consequently had to be imposed from outside. In the words of Hume, "To form an idea of an object, and to form an idea simply is the same thing; the reference of the idea to an object being an extraneous denomination, of which in itself it bears no mark or character."[50]

Bolzano's decision to look upon ideas as parts of propositions shifts the logician's focus. First of all, by broadening the traditional conception, he shows us that not all ideas (i.e. parts of propositions) serve to represent objects. For concepts such as those expressed by "and", "not", etc., are also ideas on his dispensation, and he maintains, as Wittgenstein would later on, that such words do not stand for anything, so that

[49]"*repraesentatio* [...] ist diejenige Bestimmung der Seele, die sich auf andere Dinge beziehet. Beziehen aber nenne ich, wenn ihre Beschaffenheit der Beschaffenheit der aüßern Dinge Gemäß ist, *sive si rebus externis conformis est*." One recalls at this point Kant's argument (*Critique of Pure Reason*, B40) to the effect that space must be an intuition because it is represented as having infinitely many parts, while concepts have only finitely many—if we can infer the presence of infinitely many parts in a representation on the basis of its object having infinitely many parts, it must be assumed that a representation has at least as many parts as its object.

[50]*A Treatise on Human Nature*, I. i. 7.

the question of resemblance is nonsensical. At least for such ideas, then, the traditional approach has nothing to offer.

Even when we speak of ideas with objects, however, the emphasis is no longer on the relation between idea and object. Instead, Bolzano looks at things in much the same way Frege would after him. An idea is a part of a proposition, and we will understand what it is once we understand the work it does in a proposition, the contribution this part makes to the truth or falsity of the whole.

Writing to Exner, Bolzano tells us that:

> A proposition is true when it attributes to a subject a predicate that it possesses, or (in other words) when every object that stands under the subject concept of the proposition has an attribute that stands under the predicate concept.[51]

Consider now the proposition "Czechs are Europeans." This proposition is true according to the definition because everything that stands under the concept "Czech" is a European. The contribution of the idea "Czech" as it occurs here is exhausted by the function of representing or, in linguistic terms, referring to, a certain set of people. It furnishes an extension. This is what the idea does, and for Bolzano, this is in large measure what the idea is. Once we have determined the extension of an idea, we have said just about all that can be said about it.

In certain cases, to be sure, we can explain why an idea has the extension it does in terms of the extensions of the parts of the idea. The extension of "female violinist", for example, is the intersection of the extensions of the ideas "female" and "violinist". But we are then faced with accounting for why these part-ideas have the extensions they do and thus no closer to our goal. Since, for Bolzano, complex ideas can only exist if there also exist simple ideas,[52] we can see why he maintained that "representing" was indefinable. "Why does a given simple idea have the extension it does?" is an odd sort of question for him, for simple ideas are *individuated* by their extensions.[53] A simple idea has the extension it does because otherwise it wouldn't be that idea, but some other. To ask for an explanation here is just misguided. In particular, as he explained to Exner in 1843, one should not fall into the trap of

[51] Bolzano to Exner, 18 December, 1834; below, p. 167.
[52] *Method*, §3; **WL**, §61.
[53] **WL** §92.

thinking that representation can be explained by appealing to any kind of resemblance between an idea and its object(s).

> [W]however compares this *being represented* with a kind of *depiction*; whoever wants to speak as if the object were related to its idea in roughly the same way that a *sensible* object is related to its *picture*; whoever merely assumes that there is a certain *similarity* between *idea* and *object*, an *agreement* between their respective attributes; especially, however, whoever (like our Philosophers of Identity) speaks of the complete *equality*, indeed *identity* of the two; such a person, I say, has already embarked upon the most dangerous path, one which, should he continue to wander along it, will keep his back forever turned to clear and distinct thought. For tell me what kind of *similarity* obtains, or could obtain, between the idea *something* on the one hand, and its objects on the other, i.e. each and every thing there is? Show me a pair of things *more dissimilar* than the ideas *attribute, spatial object, tool, concept, proposition, etc.* and the objects that are, respectively, represented by these ideas.[54]

Bolzano recognized that the appeal to similarity was futile, because similarity is ubiquitous (one could also remark that the relation of similarity is an equivalence relation, while that of representing, presumably, is not). "[I]t is a well known theorem that any two things, no matter how different, nevertheless have something in common, e.g. that they are both *things*. If we call a pair of things heterogeneous this is always to be understood only relatively and as a matter of degree—that they differ from each other in this or that respect, and perhaps even in many respects."[55] Appealing to similarity to explain reference thus fails because it explains too much—with "similarity" taken in a general sense, everything would have to be said to represent everything else.

If, on the other hand, one tries to appeal to a narrower notion of similarity, one encounters a different set of problems. If, Bolzano argues in the *Theory of Science*, an idea is said to be a picture [*Bild*] of its

[54]Below, p. 183.

[55]Bolzano to Exner, 18 December 1834; below, p. 163.

object, then the proposed definition is too wide. For one object is a picture of another, according to him, "if they are sufficiently similar so that there are certain purposes and occasions for which it is suitable to examine the first rather than the second." But under this definition not all pictures are in the mind. We might then try to define an idea as a *mental* picture. Bolzano writes: "If one now wished to respond to the question 'what is a mind?' by saying that it is an entity with the faculty of representation, the circle in this definition would become clear as day."[56] He adds that even if we somehow manage to avoid this circle, there remains the problem that there are other things in the mind besides ideas that might also be considered pictures in the relevant sense, notably in cases of empathy—the pain I feel in contemplating the misfortune of another might well be a picture in the relevant sense of the pain he feels.

What has gone wrong here is that philosophers, perhaps under the spell of a certain theory of representation, hadn't noticed that similarity, or depiction, simply can't explain why an idea has the objects that it does. To think that it can is a disastrous mistake. An idea is not a picture, "... not an object that we examine in place of another. Rather, it is what arises in our mind when we examine the object itself." [57]

In Exner, Bolzano encountered a living and prominent proponent of the traditional theory of ideas. To begin with, Exner rejects Bolzano's notion of ideas in themselves: these, like propositions in themselves he claims to be mere abstractions. All that really exists, he maintains, are subjective ideas, which are states of consciousness, usually caused by objects distinct from the mind. What are called objective ideas (Exner also speaks of "ideas considered logically") are just these same subjective ones considered apart from the psychological context in which they occur.

Concerning the nature of these subjective ideas, Exner informed Bolzano:

> I think of the object of an idea as a substance [...] the idea, however, as the image of this substance. As, however, the substance as such is always unknown to us, the idea is an image of its characteristics (in the widest sense of the word,

[56] **WL**, §52.6.
[57] **WL**, §52.6.

> where the relations of substances are also included), and indeed, either of all the characteristics, or some of them, or even just a single one. (Exner to Bolzano, 10 December 1834, below p. 155.)

Two points to be retained here are Exner's belief that ideas are pictures of their objects, and that what is depicted are the objects' characteristics. It is presumably the first claim that is supposed to explain what is meant by an objective idea having certain objects: this is to be interpreted as meaning that the idea is an adequate depiction of the object—it "fits" the object in a sense that remains to be specified.[58] At the same time, Exner denies that any idea has an intrinsic relation to any object, even to those that it "fits". Whereas Bolzano held the extension of an idea to be one of its internal attributes, Exner claims that "no idea in the consciousness of a man has its object in it. Rather, it must be applied to it by the subject."[59] Apparently, this application can be performed ineptly, or not at all: "In my opinion, children originally apply such states of consciousness [sc. ideas] to nothing at all; adults often apply them to the wrong objects and must afterwards correct themselves."[60]

The second point—that all ideas are composed of ideas of characteristics—is used by Exner to support the claim that all ideas, considered objectively, are general, a characteristic being by definition something that can be found in more than one thing. Exner seems to think of ideas as mental states which can occur repeatedly (by this he seems to mean that we can have qualitatively indistinguishable mental states at different times)—this, and not the intrinsic attribute of representing certain objects, is the sense of generality that Exner seems to appeal to.

With this understanding of ideas, Exner cannot fathom how Bolzanian intuitions could exist. He admits, not without some hesitation, that there might be simple ideas, but will not accept that such ideas could, considered objectively, represent only a single object:

> Suppose that I were to perceive an object, and had a simple idea of it, which therefore consisted of only one characteristic. In intuition, i.e. psychologically, this characteristic

[58]The difficulties with this view are discussed in Bolzano's 1843 essay, below, p. 176-184.

[59]*Ibid.*

[60]Exner to Bolzano, 11 November 1834; below, p. 137.

> would only be applied to a single object, yet I cannot call it a singular idea. For if it is considered logically, it surely appears as the sort of idea that can be present as a characteristic in many ideas; i.e. logically, it appears as a general idea.[61]

The subject might well apply a simple idea to a single object, but the relation to this object is due entirely to the application:

> Supposing that in a certain soul merely the idea a were present, it would apply to no object at all. For this subject it would be an objectless idea [...]. For I should not say "I represent a," rather "I have the state [of consciousness] a." This state will, however, be caused, aroused, as we assume, by an object b. It cannot, however, be doubted that exactly the same subjective state a might have been aroused by another object c or d.[62]

Overlooking all the familiar problems of Exner's sort of account, Bolzano patiently explains to his friend that he has the semantics all wrong. Consider the application of ideas that Exner appeals to. Bolzano grants that we can apply the idea "red" to many things, even to things that are not red. But this application is surely accomplished by means of judgments. Bolzano now asks: what are the terms of these judgments? When I apply the idea "red" to an object, the idea "red" can only be the predicate-, never the subject-idea. For when we apply the idea "red" to something, we judge "this object is red" or "such and such object is red", rather than "red is thus and so." This is already clear from the fact that "red" is a general idea, while the topic of our judgment is a single thing. Perhaps Exner is confused, Bolzano suggests, because he has failed to distinguish the sensation (which is a single event) from the general concept "red" under which it stands. We can indeed have many sensations that *stand under* the general idea "red", and to which we apply the idea "red". But these sensations are not ideas, nor are they represented in the judgments of application by the idea "red". Rather they are represented by singular ideas, namely either by intuitions or by

[61] Exner to Bolzano, 27 June 1833; below, p. 88.

[62] Exner to Bolzano, 11 November, 1834; below, p. 137.

ideas containing intuitions—there simply being no other way to account for singular reference in such cases.

It is clear from the letters translated here that Exner had not the slightest inkling of the semantic layer which is the entire focus of Bolzano's research, and, one might add, a basic part of the modern approach to logic. No wonder he and others had such great difficulty understanding Bolzano's innovations. No wonder too that Brentano had to "rediscover" intentionality when opinions like Exner's were commonplace and all but unchallenged, and when Bolzano's were met for the most part with indifference or blank incomprehension. For clearly even subjective ideas in Exner's sense utterly lack any such intrinsic relation to an object. Indeed, as the 1843 exchange shows, Exner was, rightly, worried that with the concepts at his disposal he might not be able to explain how an idea could bear any relation at all to an object. Speaking of objects distinct from ideas seemed to run the risk of falling into absurdity. It was not for nothing that idealism seemed such an obvious option at the time.